On minimal Legendrian submanifolds of Sasaki-Einstein manifolds

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Outline

Sasakian manifolds

Definition

(M, g) Riemannian is *Sasakian* if $C(M) = M \times \mathbb{R}^+$ with warped metric $\overline{g} = r^2 g + dr^2$ is Kähler $(C(M), \overline{g}, J)$.

Tensors on M:

- $\eta = d^c \log r|_{r=1}$ is a contact form with contact distribution $D = \ker \eta$;
- $\xi = Jr\partial_r \in \Gamma(TM)$ is its Reeb field $(\eta(\xi) = 1, \iota_\xi d\eta = 0)$;
- $TM = D \oplus L_\xi$;
- $\Phi = \begin{cases} J|_D & \text{on } D \\ 0 & \text{on } \xi \end{cases}$

It holds $\Phi^2 = -\,\mathrm{id} + \xi \otimes \eta$ and $g(\Phi\cdot, \Phi\cdot) = g + \eta \otimes \eta$;
M endowed with 4-ple (g, η, ξ, Φ) is *normal contact metric*.
Then

- $(D, \Phi|_D)$ is a CR structure on M;
- $d\eta = g(\Phi\cdot, \cdot)$ and $(D, \Phi|_D, d\eta)$ is a *transverse Kähler structure* on M with metric $g^T = g|_{D \times D}$.

The Kähler sandwich

$(C(M), \overline{g})$ Kähler cone

(M, g) Sasakian

g^T transverse Kähler metric

Theorem

g is Sasaki-Einstein iff g^T is Kähler-Einstein iff $\overline{g}$ is Ricci-flat.

Example: standard sphere

$M = S^{2n+1} \subset \mathbb{C}^{n+1}$ with

$$\eta = y_j dx_j - x_j dy_j|_{S^{2n+1}}$$
$$\xi = y_j \partial_{x_j} - x_j \partial_{y_j}|_{S^{2n+1}}$$
$$\Phi = \begin{cases} J|_{\ker \eta} & \text{on } \ker \eta \\ 0 & \text{on } \xi \end{cases}$$
$$g = \text{round metric}$$

Riemannian submersion onto Kähler manifold (space of leaves of ξ)

$$\begin{array}{c} (S^{2n+1}, g) \\ S^1 \Big\downarrow \\ (\mathbb{CP}^n, g_{\mathrm{FS}}) \end{array}$$

Example of *regular* Sasakian manifold

Minimal Legendrian submanifolds

Definition

Let (M^{2n+1}, η) be contact. A *Legendrian* submanifold is a n-dimensional submanifold $i : L \hookrightarrow M$ such that $i^*\eta = 0$.

Lê and Wang have characterized the minimal Legendrian submanifolds of S^{2n+1}.
$L^n \subset S^{2n+1}$ minimal submanifold, $M \in \mathfrak{su}(n+1)$ and $f_M(x) = \langle Mx, Jx \rangle$ as function on $L \subset S^{2n+1} \subset \mathbb{C}^{n+1}$.
They prove:

Theorem (Lê-Wang, 2001)

L is Legendrian iff f_M is an eigenfunction of $\Delta_L = \delta d$ with eigenvalue $2n+2$, which has multiplicity $\geq \frac{1}{2}n(n+3)$. Moreover if multiplicity $= \frac{1}{2}n(n+3)$ then L is totally geodesic in S^{2n+1}.

They use very specific arguments for minimal submanifolds of spheres.

Main result

We prove a partial generalization of Lê-Wang for η-Sasaki-Einstein manifolds.

Definition

(M, η, g) Sasakian is η-Sasaki-Einstein if there exists $A \in \mathbb{R}$ s.t.

$$\mathrm{Ric}_g = Ag + (2n - A)\eta \otimes \eta.$$

Main result

Let $\mathfrak{g} \neq \langle \xi \rangle$ be the infinitesimal Sasakian automorphism algebra (contactomorphic Killing fields)
L^n minimal submanifold.
For $X \in \mathfrak{g}$ consider the functions on L

$$f_X = \eta(X) - \frac{1}{\mathrm{vol}(L)} \int_L \eta(X) dv.$$

Can be seen as contact moment map for the action of Sasaki transformations.
We prove

Theorem (Calamai, –)

Let M be η-S-E and L is minimal Legendrian.
Then $\Delta_L f_X = (A+2) f_X$ and

$$m_L(A+2) \geq \dim \mathfrak{g} - \frac{1}{2} n(n+1) - 1.$$

For the sphere: $\mathfrak{g} = \mathfrak{u}(n+1) \ni Y$ and $\langle Yx, Jx \rangle = \eta(Y)|_x$.

Main result

We also prove the following rigidity result, in the *regular case*.

Theorem (Calamai, –)

M is a regular S-E manifold, L minimal Legendrian and

$$m_L(2n+2) = \dim \mathfrak{g} - \frac{1}{2}n(n+1) - 1.$$

Then L is totally geodesic in M, which is a Sasaki-Einstein circle bundle over $\mathbb{CP}^n$ with Fubini-Study metric. In particular if M is simply connected then $M = S^{2n+1}$.

Geometric remark

For $X \in \mathfrak{g}$ the map $\eta(X)$ is the *contact moment map* for the $\mathrm{Aut}(M)$-action.
In general if G acts by contactomorphisms on (M, η) we can extend the action to the symplectization $(C(M), d(r^2\eta))$ by $g(r, p) = (r, gp)$.
G acts on $C(M)$ in a Hamiltonian fashion with moment map $\varphi : C(M) \to \mathfrak{g}^*$ that can be taken to be $X \mapsto r^2\eta(X)$.
The restriction $\varphi|_{\{r=1\}}$ is called the *contact moment map*.

Minimal Legendrian submanifolds

$i : L \hookrightarrow M$ be Legendrian in a Sasakian manifold

Proposition (Ono)

There is an isomorphism

$$\chi : \Gamma(NL) \longrightarrow C^\infty(L) \oplus \Omega^1(L)$$
$$V \longmapsto \left(\eta(V), -\frac{1}{2} i^*(\iota_V d\eta)\right)$$

If M is regular over a Kähler base (B, ω) with projection π then we have the well known

Proposition (Reckziegel)

$L \subset M$ is Legendrian iff $\widetilde{L} = \pi(L)$ is Lagrangian ($(\pi \circ i)^*\omega = 0$) and $\pi : L \to \widetilde{L}$ is a finite cover.
Moreover L is minimal or totally geodesic iff $\widetilde{L}$ is.

Deformations of minimal Legendrian submanifolds

Let $i : L \hookrightarrow M$ be minimal Legendrian.

Definition

A smooth family of minimal Legendrian immersions $i_t : L \to M$ is a family of maps $F : [0,1] \times L \to M$ such that for each t the map $i_t = F(t, \cdot) : L \to M$ is a minimal Legendrian immersion.

Every smooth family points out a vector field W_t on L given at p by

$$W_t|_p = F_* \left(\frac{\partial}{\partial t} \bigg|_{(t,p)} \right).$$

Infinitesimal Legendrian deformations:

Proposition

A family of immersions is Legendrian if and only if the normal component V_t of W_t satisfies

$$V_t = \chi^{-1} \left(\eta(V_t), \frac{1}{2} d\eta(V_t) \right)$$

Deformations of minimal Legendrian submanifolds

For η-Sasaki-Einstein manifolds we can describe the whole space of infinitesimal deformations

Proposition (Ohnita)

Let $i : L \to M$ be a minimal Legendrian submanifold in an η-Sasaki-Einstein manifold with constant A. Then the vector space of infinitesimal minimal Legendrian deformations is identified with

$$\mathrm{Def}(L) = \mathbb{R} \oplus \{f \in C^\infty(L) : \Delta_L f = (A+2)f\}$$

where Δ_L denotes the Laplacian of L with the induced metric.

One-parameter family $\varphi_t \subset \mathrm{Aut}(M)$ gives minimal Legendrian deformation

$$i_t = \varphi_t|_{i(L)} : i(L) \to M,$$

called *trivial*.
Infinitesimally are given by $X^\perp$ with $X \in \mathfrak{aut}(M)$.

Proof of Theorem 1

Proof of Ohnita's prop follows from two facts:

1. Infinitesimal minimal deformations are parameterized by $\ker \mathcal{J}$, where $\mathcal{J}$ = Jacobi operator from Riemannian geometry.
2. Infinitesimal Legendrian deformations are parameterized by $C^\infty(L) \simeq \left\{ \left(f, \frac{1}{2} df \right) : f \in C^\infty(L) \right\}$

Proof of Theorem 1

1. Take $\mathfrak{g} \ni X = X_1 + X_2 \in \Gamma(TL) \oplus \Gamma(NL)$.
2. X_2 defines trivial minimal Legendrian deformation
3. $\chi(X_2) = (\eta(X), \alpha_X) \in \chi(\ker \mathcal{J})$
4. So $\Delta_L \eta(X) - (A+2)\eta(X) = \text{const} \Longrightarrow$ f_X is eigenfunction of eigenvalue $A+2$.

Proof of Theorem 1

Proof of multiplicity assertion

We have said that families of ambient transformations give trivial deformations:

$$\alpha : \mathfrak{g} \to \mathrm{Def}(L)$$

and $\ker\alpha = \{X \in \mathfrak{g} : X|_L \in \Gamma(TL)\} \subset \mathfrak{iso}(L)$ So

$$\begin{aligned} 1 + \dim E_{A+2} &\geq \dim \alpha(\mathfrak{g}) \\ &= \dim \mathfrak{g} - \dim \ker \alpha \\ &\geq \dim \mathfrak{g} - \dim \mathfrak{so}(n+1) \\ &= \dim \mathfrak{g} - \frac{n(n+1)}{2}. \end{aligned}$$

Proof of Theorem 2

Let $\pi : M \to B$ be regular Sasaki-Einstein onto Kähler-Einstein.
Well known fact: M is S-E iff B is K-E with constant $2n+2$.

1. Equality $\implies$ equality above;
2. Regularity $\implies$ $\pi(L) \subset B$ is Lagrangian with large isometry (actually homogeneous) K/H with $\mathfrak{k} = \mathfrak{so}(n+1)$;
3. K acts on B by with cohomogeneity one with two singular orbits: Kp (Lagrangian) and Kq;
4. Homogeneous Lagrangians in K-E (Bedulli-Gori) $\implies$ $\Omega = K^{\mathbb{C}}p$ is open Stein and that its complement has complex codim 1 and is another single $K^{\mathbb{C}}$-orbit (B is two-orbit Kähler);
5. Akhiezer classification of 2-orbit Kähler $\Omega \cup A$ with Ω affine and A complex hypersurface $\implies$ $\mathbb{RP}^n \subset \mathbb{CP}^n$ or $S^n \subset Q_n$;

Proof of Theorem 2

6. Explicit computation of multiplicity to exclude hyperquadrics:
   - One possible way to have $S^n \subset Q_n$ (Chen-Nagano)
   - Compute induced metric on S^n from K-E metric on Q_n ($\frac{n}{2n+2} \times$ round metric);
   - The multiplicity of the eigenvalue $2n+2$ does not attain lower bound!
7. $\pi(L) \subset B$ totally geodesic $\Longleftrightarrow L \subset M$ totally geodesic.

Other family of eigenfunctions

View $M = \{r = 1\} \subset M \times \mathbb{R}^+$ and let
$\mathfrak{k} =$ infinitesimal Kähler automorphisms of cone
L minimal Legendrian
For $K \in \mathfrak{k}$ construct function on L

$$h_K = \overline{g}(M_K \partial_r, J\partial_r)$$

where $M_K = \overline{\nabla} K + \frac{1}{2n+2} \operatorname{div}(JK) J$.
Then we prove

Theorem (Calamai, –)

M Sasaki-Einstein, L minimal Legendrian, then

$$\Delta_L h_K = (2n + 2) h_K .$$

This family generalizes previous f_X as $\mathfrak{g} \subset \mathfrak{k}$.

Other family of eigenfunctions

Lê-Wang	Our case
$f_M(x) = \langle Mx, Jx \rangle$	$h_K(p) = \overline{g}(M_K\partial_r, J\partial_r)\|_{(p,1)}$
$\langle \cdot, \cdot \rangle$	$\overline{g}$
point $\longleftrightarrow$ position vector	$p \in M \longleftrightarrow \partial_r\|_{(p,1)} \in TC(M)$
$\xi_x = Jx$	$\xi_p = J\partial_r\|_{(p,1)}$
$Mx \in \mathbb{C}^{n+1}$	$M_K\partial_r \in TC(M)$

Other family of eigenfunctions

- For S^{2n+1} the cone is $\mathbb{C}^{n+1} \setminus \{0\}$
- Take $M \in \mathfrak{su}(n+1)$
- Linear field $K : x \mapsto Mx$ is Killing and holomorphic
- Then $\overline{\nabla}_y K = My$ and $\operatorname{div}(JK) = 0$,

So $M_K(y) = My$ and

$$\overline{g}(M_K \partial_r, J\partial_r)|_{(y,1)} = \langle My, Ky \rangle.$$

Proof of result

Legendrian implies that a local ON frame $\{e_i\}$ of L can be extended to an ON frame $\{\frac{1}{r}e_i, \frac{1}{r}Je_i, \frac{1}{r}\xi, \partial_r\}$ of $C(M)$;

Minimality implies that intrinsic Laplacian = extrinsic Laplacian

$$\Delta_L f = -\sum_i \nabla df(e_i, e_i)|_L - H \cdot f|_L$$

$\overline{g}$ Ricci-flat implies that M_K has similar properties of the M of Lê-Wang.

Proof of result

$$f : (p, r) \longmapsto \overline{g}(M_K \partial_r, J\partial_r)$$
$$M_K = \overline{\nabla} K + \frac{1}{2n+2} \operatorname{div}(JK) J$$

Step 1 M_K is skew-symmetric, $\operatorname{tr} JM_K = 0$ and $\overline{\nabla} M_K = \overline{\mathrm{Rm}}(\cdot, K)$;

Step 2 $\overline{\mathrm{Rm}}(\cdot,\cdot,\cdot,\cdot)$ vanishes along certain directions;

Step 3 Finally compute

$$\begin{aligned} \Delta_L f|_L &= -\nabla df(e_i, e_i)|_L = -(e_i \cdot e_i \cdot f - \nabla_{e_i} e_i \cdot f)|_L \\ &= (2n+2) f|_L \end{aligned}$$

Open questions

1. Let M Sasakian with big enough automorphism group and L minimal with f_X or h_K as eigenfunctions of the Laplacian. Can we conclude that L is Legendrian?
2. Do totally geodesic L attain lower bound?
3. Remove regularity assumption in rigidity result;
4. Give a geometric meaning to the family $\{h_K : K \in \mathfrak{k}\}$.

Thank you!