A Poincaré-Bendixson theorem for meromorphic connections on Riemann surfaces

Joint work with Marco Abate

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Levico Terme, October 20, 2014

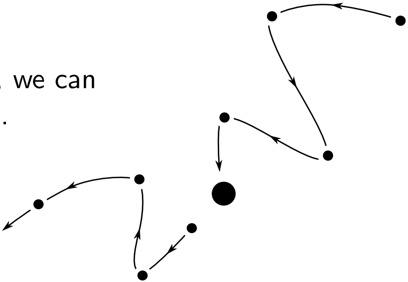
A long time ago...

... in a part of mathematics (seemingly) far far away...

Local holomorphic dynamics

Study of the iterations of a holomorphic endomorphism f of a complex manifold M in a neighbourhood of a fixed point z_0 for f.

Up to working in a suitable chart, we can suppose that $M = \mathbb{C}^n$ and $z_0 = 0$.



Dynamics in $\mathbb{C}$

$f(z) = \lambda z + O(z^2)$

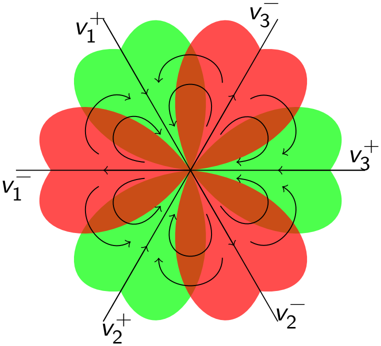
$$\begin{aligned} |\lambda| < 1 \quad &\Leftrightarrow \quad 0 \text{ attracting} \\ |\lambda| = 1 \quad &\Leftrightarrow \quad 0 \text{ indifferent} \\ |\lambda| > 1 \quad &\Leftrightarrow \quad 0 \text{ repelling} \end{aligned}$$

$$\text{Indifferent} \Rightarrow \begin{cases} \lambda = e^{2\pi i \alpha}, \alpha \in \mathbb{R} \setminus \mathbb{Q} & \text{irrational} \\ \lambda = e^{2\pi i p/q}, p/q \in \mathbb{Q} & \text{rational or } \textit{parabolic} \end{cases}$$

Parabolic case: up to considering an iterate, we can suppose that f is *tangent to the identity*, i.e., that $\lambda = 1$

$$f(z) = z + a_{\nu+1} z^{\nu+1} + \dots, \qquad a_{\nu+1} \neq 0$$

Leau-Fatou flower theorem- 1920



$$f(z) = z + a_{\nu+1} z^{\nu+1} + \dots, \qquad a_{\nu+1} \neq 0$$

Theorem (Camacho 1978, Scherbakov 1982)

Let $f(z) = z + a_{\nu+1}z^{\nu+1} + \dots$, $a_{\nu+1} \neq 0$, be a germ of a holomorphic function tangent to the identity. Then f is locally topologically conjugated to the time-1 map of the holomorphic homogeneous vector field

$$Q = z^{\nu+1}\frac{\partial}{\partial z}$$

$$z \mapsto \frac{z}{(1-\nu z^{\nu})^{1/\nu}} = z + z^{\nu+1} + \dots$$

Maps tangent to the identity in $\mathbb{C}^n$

$$f(z) = z + Q_{\nu+1}(z) + \dots$$

$$Q_{\nu+1} = (Q^1, \dots, Q^n)$$

Q^j homogeneous polynomials of degree $\nu + 1$

Goal: description of the dynamics in a full neighbourhood of the origin.

Maps tangent to the identity in $\mathbb{C}^n$ (1985 - now)

New phenomena: for example, orbits converging to the origin without being tangent to any direction.

Results: Écalle, Hakim, Weickert, Abate, ecc.

Abate-Tovena [AT2011]

Description of the dynamics in a full neighbourhood of the origin for an important class of examples.

A new hope

Theorem (Camacho 1978 - Scherbakov 1982)

Let $f(z) = z + a_{\nu+1}z^{\nu+1} + \dots$, $a_{\nu+1} \neq 0$, be a germ of a holomorphic function tangent to the identity. Then f is locally topologically conjugated to the time-1 map of the holomorphic homogeneous vector field

$$Q = z^{\nu+1}\frac{\partial}{\partial z}$$

Conjecture

A similar statement holds also in $\mathbb{C}^n$, for *generic* maps tangent to the identity.

Study of holomorphic homogeneous vector fields

$$Q = Q^1 \frac{\partial}{\partial w^1} + \cdots + Q^n \frac{\partial}{\partial w^n}$$

homogeneous field of degree $\nu + 1$

$$f_Q(z) = z + Q_{\nu+1}(z) + \dots$$

the time-1 map, where

$$Q_{\nu+1} = (Q^1, \dots, Q^n)$$

Study of holomorphic homogeneous vector fields

$[v] \in \mathbb{P}^{n-1}$ *characteristic direction*: $L_{[v]} = \mathbb{C}v$ is Q-invariant (*degenerate* if $Q = 0$ there)

The dynamics is 1-dimensional.

Hakim - 1998

If $f^k(z) \to 0$ tangent to $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$, then $[v]$ is characteristic.

Q *dicritical*: multiple of the radial field, all directions are characteristic

We study the dynamics of *non dicritical* homogeneous vector field *outside the characteristic leaves*

Meromorphic connections

Idea: derive meromorphic sections of the tangent bundle

Definition

A *meromorphic connection* ∇ on the tangent bundle of a Riemann surface $TS \to S$ is a $\mathbb{C}$-linear map

$$\mathcal{T}\mathcal{S} \to \mathcal{M}_S^1 \otimes \mathcal{T}\mathcal{S}$$

that satisfies the Leibniz rule $\nabla(fe) = df \otimes e + f\nabla e$.

- Locally, on $(U_\alpha, \partial_\alpha)$, we have $\nabla\partial_\alpha = \eta_\alpha \otimes \partial_\alpha$, for some meromorphic η_α
- We define the *residue* $\mathrm{Res}_p(\nabla) := \mathrm{Res}_p(\eta_\alpha)$
- A *geodesic* is a curve $\gamma : I \to S^0$ such that $\nabla_{\sigma'}\sigma' \equiv 0$

The Empire strikes back

Theorem (Abate-Bracci-Tovena 2004, Abate-Tovena 2011)

Let Q be a non-dicritical holomorphic homogeneous vector field and $\hat{S}_Q \subset \mathbb{C}^n$ the complement of the characteristic leaves.
There exists a foliation of $E \cong \mathbb{P}^{n-1}$ in Riemann surfaces and a partial meromorphic connection ∇ on E such that, for $\gamma : I \to \hat{S}_Q$ the following are equivalent:

- γ is an integral curve for Q in $\mathbb{C}^n$;
- $[\gamma]$ is a geodesic for ∇ on (a leaf of) E

Remark

The poles of ∇ are the characteristic direction of Q

- Study γ
  - Study $[\gamma]$
    - Study the foliation in Riemann surfaces
    - Behaviour of $[\gamma]$ in a leaf
  - Recover γ from $[\gamma]$

The return of the jedi

$\gamma : [0, \varepsilon) \to \mathbb{C}^n$

- Asymptotic study of $[\gamma(t)]$
- Classification of the possible ω-*limits*

Definition

$p \in \omega([\gamma])$ if there exist $t_n \to \varepsilon$ with $[\gamma(t_n)] \to p$

$$\omega([\gamma]) = \bigcap_{\varepsilon' < \varepsilon} \overline{\{[\gamma(t)] : t > \varepsilon'\}}$$

Poincaré-Bendixson theorems

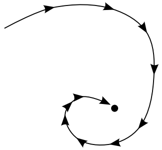
Description of the possible ω-limits

- Study γ
  - Study $[\gamma]$
    - Study the foliation in Riemann surfaces
    - Behaviour of $[\gamma]$ in a leaf
      - **Study the ω-limits of geodesics for meromorphic connections on Riemann surfaces**
  - Recover γ from $[\gamma]$

Theorem ([AT 2011] for $S = \mathbb{P}^1(\mathbb{C})$, Abate-B. 2014)

Let ∇ be a meromorphic connection on the tangent bundle of a compact Riemann surface S, let S^0 be the complement of the poles and $\sigma : [0, \varepsilon) \to S^0$ a maximal geodesic for ∇. Then, for $t \to \varepsilon$,

1. $\sigma(t)$ tends to a pole; or
2. σ is closed; or
3. σ tends to the support of a closed geodesic; or
4. σ accumulates a graph of *saddle-connections*; or
5. σ self-intersects ∞ many times; or
6. $\mathring{\omega}(\sigma) \neq \emptyset$.



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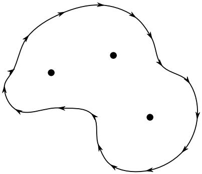

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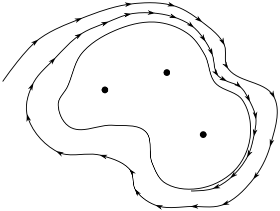

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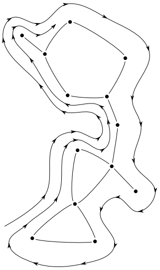

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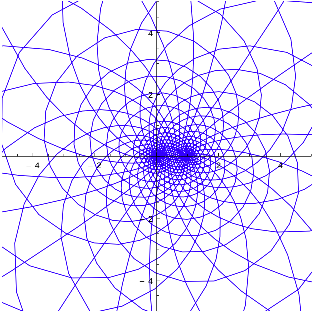

$t = 400$

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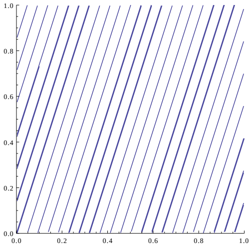

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If $\omega(\sigma)$ *disconnects*

$$\sum_{p_j \in P} \mathrm{Re}\,\mathrm{Res}_{p_j}(\nabla) = -\chi_P$$

Notice that

$$\sum_{p_j} \mathrm{Res}_{p_j}(\nabla) = -\chi_S$$

Sketch of proof - The local geometry (AT2011)

Let U be a local chart of S^0, with U simply connected. Then:

- locally, metric induced by ∇: geodesics for ∇ are geodesics for this metric;
- explicit local isometry J from U to an open subset of $\mathbb{C}$, endowed with the Euclidean metric;
- geodesic segments correspond to euclidean segments;
- locally, explicit form for the geodesics.

Sketch of proof - Extension

Lemma

Let S be a Riemann surface and ∇ a meromorphic connection on S, with poles $p_1, \dots, p_r \in S$. Let $S^0 = S \setminus \{p_1, \dots, p_r\}$.

Let $\sigma \colon I \to S^0$ be a geodesic for ∇ without self-intersections, maximal in both forward and backward time.

Then there exists a smooth line field Λ with singularities on S such that

- σ is an integral curve for Λ;
- in a neighbourhood of σ, Λ is singular exactly on the poles of ∇.

Furthermore, the ω-limit W of σ is Λ-invariant, and $\Lambda|_W$ is uniquely determined.

Sketch of proof - Minimal sets for Λ

A *minimal set* for a line field Λ is a closed, non-empty, Λ-invariant subset of S without proper subsets having the same properties.

Hounie 1981

Let S be a compact connected two-dimensional smooth real manifold (e.g., a Riemann surface) and let Λ be a smooth line field with singularities on S. Then a Λ-minimal set Ω must be one of the following:

1. a singularity of Λ;
2. a closed integral curve of Λ, homeomorphic to S^1;
3. all of S ($\Rightarrow$ torus).

Sketch of proof - Residues

Gauss-Bonnet-like theorem

Let ∇ be a meromorphic connection on a compact Riemann surface S, with poles $\{p_1, \ldots, p_r\}$ and set $S^0 := S \setminus \{p_1, \ldots, p_r\}$.

Let P be a subset of S whose boundary is given by m geodesic cycles, positively oriented with respect to P. Let $z_1, \ldots, z_s$ denote the vertices of the geodesic cycles, and $\varepsilon_j \in (-\pi, \pi)$ the external angle at z_j. Suppose that P contains the poles $\{p_1, \ldots, p_g\}$ and denote by $g_{\hat{P}}$ the genus of the *filling* $\hat{P}$ of P. Then

$$\sum_{j=1}^{s} \varepsilon_j = 2\pi \left(2 - m - 2g_{\hat{P}} + \sum_{j=1}^{g} \operatorname{Re} \operatorname{Res}_{p_j}(\nabla) \right) .$$