Cohomological properties of symplectic manifolds

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Kähler manifolds have special cohomological properties,

A. Weil, *Introduction à l'etude des variétés kählériennes*, Hermann, Paris, 1958.

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both from complex. . .

Sur une variété compacte V *de type kählérien,*

Théorème 3. — *L'espace de cohomologie* $\mathscr{H}$(V) *d'une variété compacte* V *de type kählérien est somme directe des espaces* $\mathscr{H}^{a,b}$(V).

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...and from symplectic point of view.

THÉORÈME 5. — *Soient V une variété compacte de type kählérien de dimension complexe n, et* **u** *une classe de cohomologie de type kählérien sur V. Alors toute classe de cohomologie* **a** *de degré p sur V peut se mettre, d'une manière et d'une seule, sous la forme*

$$\mathbf{a} = \sum_{r \geqslant (p-n)^+} \mathrm{L}^r \mathbf{a}_r \quad , \tag{III}$$

Interest on non-Kähler manifold since 70s...

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SOME SIMPLE EXAMPLES OF SYMPLECTIC MANIFOLDS

W. P. THURSTON

ABSTRACT. This is a construction of closed symplectic manifolds with no Kaehler structure.

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SOME SIMPLE EXAMPLES OF SYMPLECTIC MANIFOLDS

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. . . until now.

Generalized Cohomologies and Supersymmetry

Li-Sheng Tseng[1], Shing-Tung Yau[2]

Providing a full accounting of all the massless moduli from geometry will necessitate a deeper understanding of non-Kähler geometry than what is currently available. In this paper, we have given yet another example that the mathematical tools involved in non-Kähler flux compactifications, in particular here cohomologies, are generally not identical to those in Kähler geometry and Calabi-Yau compactifications. As geometries that are non-Kähler are much more diverse and flexible than that of Kähler Calabi-Yau, one expects that more refined tools will be required to characterize them. Developing them will certainly help us gain deeper insights into vast regions of the still mysterious landscape of supersymmetric flux vacua.

Aim:

- study cohomology decompositions on symplectic manifolds,
- taking inspiration from the complex case
- and framing into generalized-complex geometry.
- Special classes of manifolds provide explicit examples.

Brylinski's "Hodge theory" for symplectic manifolds:

Let X be a cpt symplectic manifold.

J.-L. Brylinski, A differential complex for Poisson manifolds, *J. Differ. Geom.* 28 (1988), no. 1, 93–114.

J.-L. Koszul, Crochet de Schouten-Nijenhuis et cohomologie, The mathematical heritage of Élie Cartan (Lyon, 1984), *Astérisque* 1985, Numero Hors Serie, 257–271.

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Let X be a cpt symplectic manifold.Consider the operators

$$\mathrm{d}\colon \wedge^\bullet X \to \wedge^{\bullet+1} X \quad \text{and} \quad \mathrm{d}^\Lambda := [\mathrm{d}, -\iota_{\omega^{-1}}] : \wedge^\bullet X \to \wedge^{\bullet-1} X$$

as the counterpart of ∂ and $\overline{\partial}$ in complex geometry.

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as the counterpart of ∂ and $\overline{\partial}$ in complex geometry.

Then

$$\left(\wedge^{\bullet} X, \mathrm{d}, \mathrm{d}^{\Lambda}\right)$$

is a bi-differential $\mathbb{Z}$-graded algebra.

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Cohomologies for symplectic manifolds:

Define the cohomologies

$$H^{\bullet}_{BC,\omega}(X) := \frac{\ker \mathrm{d} \cap \ker \mathrm{d}^{\Lambda}}{\mathrm{im}\, \mathrm{d}\, \mathrm{d}^{\Lambda}} \quad \text{and} \quad H^{\bullet}_{A,\omega}(X) := \frac{\ker \mathrm{d}\, \mathrm{d}^{\Lambda}}{\mathrm{im}\, \mathrm{d} + \mathrm{im}\, \mathrm{d}^{\Lambda}} .$$

L.-S. Tseng, S.-T. Yau, Cohomology and Hodge Theory on Symplectic Manifolds: I. II, *J. Differ. Geom.* 91 (2012), no. 3, 383–416, 417–443.

C.-J. Tsai, L.-S. Tseng, S.-T. Yau, Cohomology and Hodge Theory on Symplectic Manifolds: III, arXiv:1402.0427v2 [math.SG].

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Hodge theory applies.

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Cohomologies of symplectic manifolds, vi

symplectic cohomologies, iii

Natural maps between cohomologies:

The identity induces the maps

$$
\begin{array}{ccccc}
 & & H^{\bullet}_{BC,\omega}(X) & & \\
 & \swarrow & & \searrow & \\
H^{\bullet}_{dR}(X;\mathbb{R}) & & & & H^{\bullet}_{d^{\Lambda},\omega}(X) \\
 & \searrow & & \swarrow & \\
 & & H^{\bullet}_{A,\omega}(X) & &
\end{array}
$$

Cohomologies of symplectic manifolds, vii

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Cohomologies of symplectic manifolds, viii

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- $H^{\bullet}_{BC,\omega}$ contains informations on the symplectic struct, and
- $H^{\bullet}_{BC,\omega} \to H^{\bullet}_{dR}$ allows their comparison with top aspects.

The surjectivity of the map

$$H^{\bullet}_{BC,\omega}(X) \to H^{\bullet}_{dR}(X;\mathbb{R})$$

yields to a natural symplectic decomposition of de Rham cohom.

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yields to a natural symplectic decomposition of de Rham cohom.

It corresponds to: each de Rham class admits a d-closed d^{Λ}-closed representative:

Conjecture 2.2.7. If M is a symplectic manifold which is compact, any cohomology class in $H^*(M, \mathbf{C})$ has a representative α such that $d\alpha = \delta\alpha = 0$.

J.-L. Brylinski, A differential complex for Poisson manifolds, *J. Differ. Geom.* 28 (1988), no. 1, 93–114.

Cohomologies of symplectic manifolds, xi

cohomological decompositions, iii

Thm (Mathieu, Yan, Merkulov, Guillemin, Cavalcanti)

Let X be a compact 2n-mfd endowed with ω symplectic.
The following are equivalent:

- *Brylinski's conj:* any de Rham class has d-closed d^Λ-closed repres;
- *Brylinski's $\mathcal{C}^\infty$-fullness:* $H^\bullet_{BC,\omega} \to H^\bullet_{dR}$ surj;
- *Hard Lefschetz Condition:* $[\omega^k] \smile \cdot : H^{n-k}_{dR} \to H^{n+k}_{dR}$ isom, $\forall k$;
- *$d\,d^\Lambda$-Lemma:* $H^\bullet_{BC,\omega} \to H^\bullet_{dR}$ inj;
- *sympl cohom relation:* $H^\bullet_{BC,\omega} \to H^\bullet_{dR}$ isom;
- *Lefschetz dec in cohom:* $H^\bullet_{dR} = \bigoplus_k L^k PH^{\bullet-2k}$.

O. Mathieu, Harmonic cohomology classes of symplectic manifolds, *Comment. Math. Helv.* **70** (1995), no. 1, 1–9.

D. Yan, Hodge structure on symplectic manifolds, *Adv. Math.* 120 (1996), no. 1, 143–154.

G. R. Cavalcanti, New aspects of the dd^c-lemma, Oxford University D. Phil thesis, arXiv:math/0501406v1 [math.DG].

A weaker symplectic decomposition property:

- Lefschetz decomposition moves to de Rham cohom:

$$H^{\bullet}_{dR} = \bigoplus_{r,s} H^{(r,s)}_{\omega}(X)$$

where

$$H^{(r,s)}_{\omega}(X) := \left\{ [\alpha] \in H^{2r+s}_{dR}(X;\mathbb{R}) \;:\; \alpha \in L^r P^s \right\} .$$

—, A. Tomassini, Symplectic manifolds and cohomological decomposition, *J. Symplectic Geom.* 12 (2014), no. 2, 215–236.

Thm (—, A. Tomassini)

Let X be a cpt $2n$-mfd endowed with ω symplectic. Then

$$H^2_{dR}(X;\mathbb{R}) \;=\; H^{(1,0)}_{\omega}(X) \oplus H^{(0,2)}_{\omega}(X)\;.$$

Ch. Benson, C. S. Gordon, Kähler and symplectic structures on nilmanifolds, *Topology* 27 (1988), no. 4, 513–518.

M. Rinaldi, Proprietà coomologiche di varietà simplettiche, Tesi di Laurea, Università degli Studi di Parma, a.a. 2012/2013.

Cohomologies of symplectic manifolds, xiv

cohomological decompositions, vi

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In part, if $2n = 4$, then Lefschetz dec moves to de Rham cohom.

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But non-tori nilmanifolds does not satisfy HLC (Benson and Gordon).

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Higher-dim non-HLC mfds for which Lefschetz dec moves to de Rham cohom by Rinaldi.

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M. Rinaldi, Proprietà coomologiche di varietà simplettiche, Tesi di Laurea, Università degli Studi di Parma, a.a. 2012/2013.

Thm (—, A. Tomassini)

Let X be a $2n$-dim cpt symplectic mfd.

—, A. Tomassini, Inequalities *à la* Frölicher and cohomological decomposition, to appear in *J. Noncommut. Geom.*.

Cohomologies of symplectic manifolds, xviii

inequality *à la* Frölicher for symplectic structures, ii

Thm (—, A. Tomassini)

Let X be a $2n$-dim cpt symplectic mfd. Then, for any k,

$$\dim_{\mathbb{R}} H^k_{BC,\omega}(X) + \dim_{\mathbb{R}} H^k_{A,\omega} \;\geq\; 2\,\dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R})\;.$$

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$$\dim_{\mathbb{R}} H^k_{BC,\omega}(X) + \dim_{\mathbb{R}} H^k_{A,\omega} \;\geq\; 2\,\dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R})\,.$$

Furthermore, the equality

$$\dim_{\mathbb{R}} H^k_{BC,\omega}(X) + \dim_{\mathbb{R}} H^k_{A,\omega} \;=\; 2\,\dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R})$$

holds for any k if and only if X satisfies $d\,d^{\Lambda}$-Lemma.

—, A. Tomassini, Inequalities *à la* Frölicher and cohomological decomposition, to appear in *J. Noncommut. Geom.*.

(Generalized-)complex case:

- for compact complex manifolds: (—, A. Tomassini)

$$\begin{aligned}\dim_{\mathbb{C}} H^{\bullet}_{BC}(X) + \dim_{\mathbb{C}} H^{\bullet}_{A} &\geq \dim_{\mathbb{C}} H^{\bullet}_{\partial}(X) + \dim_{\mathbb{C}} H^{\bullet}_{\overline{\partial}}(X) \\ &\geq 2 \dim_{\mathbb{C}} H^{\bullet}_{dR}(X; \mathbb{C})\end{aligned}$$

  and equalities hold *iff* $\partial\overline{\partial}$-Lemma;

- the results can be generalized to generalized-complex structures. (—, A. Tomassini; Chan, Suen)

—, A. Tomassini, On the $\partial\overline{\partial}$-Lemma and Bott-Chern cohomology, *Invent. Math.* 192 (2013), no. 1, 71–81.

K. Chan, Y.-H. Suen, A Frölicher-type inequality for generalized complex manifolds, arXiv:1403.1682 [math.DG].

The point, in the symplectic case, is that the "associated" Frölicher spectral sequences degenerate at the first step.

J.-L. Brylinski, A differential complex for Poisson manifolds, *J. Differ. Geom.* 28 (1988), no. 1, 93–114.

M. Fernández, R. Ibáñez, M. de León, The canonical spectral sequences for Poisson manifolds, *Isr. J. Math.* 106 (1998), no. 1, 133–155.

$X = \Gamma \backslash G$ nilmanifold (compact quotients of connected simply-connected nilpotent Lie groups G by co-compact discrete subgroups Γ).

Cohomologies of symplectic manifolds, xxiii
symplectic cohomologies of nil/solv-manifolds, ii

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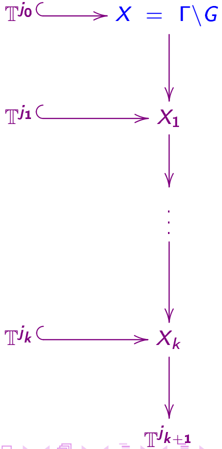

$X = \Gamma \backslash G$ nilmanifold (compact quotients of connected simply-connected nilpotent Lie groups G by co-compact discrete subgroups Γ).

Consider the finite-dim space of forms invariant for left-action $G \curvearrowright X$:

$$\iota \colon \wedge^\bullet \mathfrak{g}^* \hookrightarrow \wedge^\bullet X \ .$$

$$\begin{array}{ccc} \mathbb{T}^{j_0} & \hookrightarrow & X = \Gamma\backslash G \\ & & \downarrow \\ \mathbb{T}^{j_1} & \hookrightarrow & X_1 \\ & & \downarrow \\ & & \vdots \\ & & \downarrow \\ \mathbb{T}^{j_k} & \hookrightarrow & X_k \\ & & \downarrow \\ & & \mathbb{T}^{j_{k+1}} \end{array}$$

Thm (Nomizu)

For nilmanifolds, $H_{dR}(\iota)$ isom.

$X = \Gamma \backslash G$ solvmanifold (compact quotients of connected simply-connected solvable Lie groups G by co-compact discrete subgroups Γ).

P. de Bartolomeis, A. Tomassini, On solvable generalized Calabi-Yau manifolds, *Ann. Inst. Fourier (Grenoble)* 56 (2006), no. 5, 1281–1296.

H. Kasuya, Minimal models, formality and hard Lefschetz properties of solvmanifolds with local systems, *J. Differ. Geom.* 93 (2013), 269–298.

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In this case, de Rham cohomology may depend on the lattice.

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Thm (Kasuya)

For solvmanifolds $X = \Gamma \backslash G$, there exists a finite-dim sub-algebra

$$\iota \colon (A^\bullet, \mathrm{d}) \hookrightarrow (\wedge^\bullet X, \mathrm{d})$$

such that $H_{dR}(\iota)$ isom.

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Take $X = \Gamma / G$ nilmfd (*resp.*, solvmfd).

M. Macrì, Cohomological properties of unimodular six dimensional solvable Lie algebras, arXiv:1111.5958v2 [math.DG].

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Take $X = \Gamma/G$ nilmfd (*resp.*, solvmfd).

Suppose $\omega \in \wedge^2 \mathfrak{g}^*$ (*resp.*, $\omega \in A^2_\Gamma \cap \wedge^2 \mathfrak{g}^*$) symplectic.

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Also d^Λ-cohomology can be computed by just $\wedge^\bullet \mathfrak{g}^*$ (*resp.*, $A^\bullet_\Gamma$).

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Thm (—, H. Kasuya)

Let $X = \Gamma / G$ solvmfd with $\omega \in A^2_\Gamma$ left-inv symplectic. Then symplectic cohomologies $H^\bullet_{BC,\omega}$ and $H^\bullet_{A,\omega}$ are computed by $(A^\bullet_\Gamma, \mathrm{d})$.

M. Macrì, Cohomological properties of unimodular six dimensional solvable Lie algebras, arXiv:1111.5958v2 [math.DG].

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Generalized-complex structures:

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- cplx structure:
  $J\colon TX \xrightarrow{\cong} TX$ satisfying an algebraic condition ($J^2 = -\,\mathrm{id}_{TX}$) and an analytic condition (integrability in order to have holomorphic coordinates).

[iNδAM]

Generalized-complex structures:

- cplx structure:
  $J\colon TX \xrightarrow{\simeq} TX$ satisfying an algebraic condition ($J^2 = -\operatorname{id}_{TX}$) and an analytic condition (integrability in order to have holomorphic coordinates).
- sympl structure:
  $\omega\colon TX \xrightarrow{\simeq} T^*X$ satisfying an algebraic condition (ω non-deg 2-form) and an analytic condition ($\mathrm{d}\,\omega = 0$).

Hence, consider the bundle $TX \oplus T^*X$.

N. J. Hitchin, Generalized Calabi-Yau manifolds, *Q. J. Math.* **54** (2003), no. 3, 281–308.

M. Gualtieri, Generalized complex geometry, Oxford University DPhil thesis, arXiv:math/0401221 [math.DG].

G. R. Cavalcanti, New aspects of the dd^c-lemma, Oxford University D. Phil thesis, arXiv:math/0501406 [math.DG].

Hence, consider the bundle $TX \oplus T^*X$. Note that it admits a natural bilinear pairing: $\langle X+\xi | Y+\eta\rangle = \frac{1}{2}\left(\iota_X\eta + \iota_Y\xi\right)$.

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Mimicking the def of cplx and sympl structures:

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Mimicking the def of cplx and sympl structures:

a generalized-complex structure on a $2n$-dim mfd X is a

$$\mathcal{J}\colon TX \oplus T^*X \to TX \oplus T^*X$$

such that $\mathcal{J}^2 = -\mathrm{id}_{TX\oplus T^*X}$, being orthogonal wrt $\langle - | = \rangle$, and satisfying an integrability condition.

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Generalized-cplx geom unifies cplx geom and sympl geom:

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- J cplx struct:

$$\mathcal{J} = \left(\begin{array}{c|c} -J & 0 \\ \hline 0 & J^* \end{array}\right) \qquad \text{gen-cplx;}$$

- ω sympl struct:

$$\mathcal{J} = \left(\begin{array}{c|c} 0 & -\omega^{-1} \\ \hline \omega & 0 \end{array}\right) \qquad \text{gen-cplx.}$$

As in cplx/sympl cases, generalized-complex structures gives

$$\left(U^{\bullet},\, \partial,\, \overline{\partial}\right) \quad \text{bi-diff graded algebra}\,.$$

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$$\left(U^\bullet,\, \partial,\, \overline{\partial}\right) \quad \text{bi-diff graded algebra} \;.$$

Hence $\rightsquigarrow$ generalized-cplx cohomologies: $GH^\bullet_{\overline{\partial}}$, $GH^\bullet_{BC}$, $GH^\bullet_{A}$.

Explicit examples can be obtained by:

Thm (—, S. Calamai)

For nilmanifolds with "suitable" gen-cplx structures, generalized-complex cohomologies are computed by just left-invariant forms.

—, S. Calamai, Cohomologies of generalized-complex manifolds and nilmanifolds, arXiv:1405.0981 [math.DG].

Generalized-complex geometry, xiii
generalized-complex cohomologies, iv

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Example:

moduli-space of left-inv cplx structures on Iwasawa manifold has two connected components.

G. R. Cavalcanti, M. Gualtieri, Generalized complex structures on nilmanifolds, *J. Symplectic Geom.* 2 (2004), no. 3, 393–410.

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- In fact, the connecting gen-cplx structure is $\mathcal{C}^\infty$-full as gen-cplx (—, S. Calamai, A. Latorre).

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GeCo GeDi project:

http://gecogedi.dimai.unifi.it

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