Cohomological properties of symplectic manifolds

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[iNδAM]

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June 30, 2014

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Kähler manifolds have special cohomological properties,

A. Weil, *Introduction à l'etude des variétés kählériennes*, Hermann, Paris, 1958.

Kähler manifolds have special cohomological properties,

both from complex...

Sur une variété compacte V *de type kählérien,*

THÉORÈME 3. — *L'espace de cohomologie* $\mathscr{H}(V)$ *d'une variété compacte* V *de type kählérien est somme directe des espaces* $\mathscr{H}^{a,b}(V)$.

Introduction, iii
(non-)Kähler geometry, iii

Kähler manifolds have special cohomological properties,

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THÉORÈME 3. — *L'espace de cohomologie* $\mathscr{H}(V)$ *d'une variété compacte* V *de type kählérien est somme directe des espaces* $\mathscr{H}^{a,b}(V)$.

...and from symplectic point of view.

THÉORÈME 5. — *Soient* V *une variété compacte de type kählérien de dimension complexe* n, *et* $\mathbf{u}$ *une classe de cohomologie de type kählérien sur* V. *Alors toute classe de cohomologie* $\mathbf{a}$ *de degré* p *sur* V *peut se mettre, d'une manière et d'une seule, sous la forme*

$$\text{(III)} \qquad \mathbf{a} = \sum_{r \geq (p-n)^+} L^r \mathbf{a}_r \ ,$$

Introduction, iv
(non-)Kähler geometry, iv

Interest on non-Kähler manifold since 70s...

Introduction, v
(non-)Kähler geometry, v

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SOME SIMPLE EXAMPLES OF SYMPLECTIC MANIFOLDS

W. P. THURSTON

ABSTRACT. This is a construction of closed symplectic manifolds with no Kaehler structure.

Introduction, vi
(non-)Kähler geometry, vi

Interest on non-Kähler manifold since 70s. . .

SOME SIMPLE EXAMPLES OF SYMPLECTIC MANIFOLDS

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ABSTRACT. This is a construction of closed symplectic manifolds with no Kaehler structure.

. . . until now.

Generalized Cohomologies and Supersymmetry

Li-Sheng Tseng[1], Shing-Tung Yau[2]

Providing a full accounting of all the massless moduli from geometry will necessitate a deeper understanding of non-Kähler geometry than what is currently available. In this paper, we have given yet another example that the mathematical tools involved in non-Kähler flux compactifications, in particular here cohomologies, are generally not identical to those in Kähler geometry and Calabi-Yau compactifications. As geometries that are non-Kähler are much more diverse and flexible than that of Kähler Calabi-Yau, one expects that more refined tools will be required to characterize them. Developing them will certainly help us gain deeper insights into vast regions of the still mysterious landscape of supersymmetric flux vacua.

Introduction, vii
aim, i

Aim:

- study cohomology decompositions on symplectic manifolds,
- taking inspiration from the complex case
- and framing into generalized-complex geometry.
- Special classes of manifolds provide explicit examples.

Cohomologies of symplectic manifolds, i
Brylinski's Hodge theory, i

Brylinski's "Hodge theory" for symplectic manifolds:

Let X be a cpt symplectic manifold.

J.-L. Brylinski, A differential complex for Poisson manifolds, *J. Differ. Geom.* 28 (1988), no. 1, 93–114.

J.-L. Koszul, Crochet de Schouten-Nijenhuis et cohomologie, The mathematical heritage of Élie Cartan (Lyon, 1984), *Astérisque* 1985, Numero Hors Serie, 257–271.

Cohomologies of symplectic manifolds, ii
Brylinski's Hodge theory, ii

Brylinski's "Hodge theory" for symplectic manifolds:

Let X be a cpt symplectic manifold.Consider the operators

$$\mathrm{d}\colon \wedge^{\bullet} X \to \wedge^{\bullet+1} X \quad \text{and} \quad \mathrm{d}^{\Lambda} := [\mathrm{d}, -\iota_{\omega^{-1}}] : \wedge^{\bullet} X \to \wedge^{\bullet-1} X$$

as the counterpart of ∂ and $\overline{\partial}$ in complex geometry.

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Cohomologies of symplectic manifolds, iii
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as the counterpart of ∂ and $\overline{\partial}$ in complex geometry.

Then

$$\left(\wedge^\bullet X,\, \mathrm{d},\, \mathrm{d}^\Lambda\right)$$

is a bi-differential $\mathbb{Z}$-graded algebra.

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Cohomologies of symplectic manifolds, iv
symplectic cohomologies, i

Cohomologies for symplectic manifolds:

Define the cohomologies

$$H^\bullet_{BC,\omega}(X) := \frac{\ker \mathrm{d} \cap \ker \mathrm{d}^\Lambda}{\operatorname{im} \mathrm{d}\,\mathrm{d}^\Lambda} \quad \text{and} \quad H^\bullet_{A,\omega}(X) := \frac{\ker \mathrm{d}\,\mathrm{d}^\Lambda}{\operatorname{im} \mathrm{d} + \operatorname{im} \mathrm{d}^\Lambda} .$$

L.-S. Tseng, S.-T. Yau, Cohomology and Hodge Theory on Symplectic Manifolds: I. II, *J. Differ. Geom.* 91 (2012), no. 3, 383–416, 417–443.

C.-J. Tsai, L.-S. Tseng, S.-T. Yau, Cohomology and Hodge Theory on Symplectic Manifolds: III, arXiv:1402.0427v2 [math.SG].

L.-S. Tseng, S.-T. Yau, Generalized Cohomologies and Supersymmetry, *Comm. Math. Phys.* 326 (2014), no. 3, 875–885.

Cohomologies of symplectic manifolds, v
symplectic cohomologies, ii

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Hodge theory applies.

L.-S. Tseng, S.-T. Yau, Cohomology and Hodge Theory on Symplectic Manifolds: I. II, *J. Differ. Geom.* 91 (2012), no. 3, 383–416, 417–443.

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Cohomologies of symplectic manifolds, vi
symplectic cohomologies, iii

Natural maps between cohomologies:

The identity induces the maps

$$\begin{array}{ccccc} & & H^{\bullet}_{BC,\omega}(X) & & \\ & \swarrow & & \searrow & \\ H^{\bullet}_{dR}(X;\mathbb{R}) & & & & H^{\bullet}_{\mathrm{d}^{\Lambda},\omega}(X) \\ & \searrow & & \swarrow & \\ & & H^{\bullet}_{A,\omega}(X) & & \end{array}$$

Cohomologies of symplectic manifolds, vii

symplectic cohomologies, iv

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Cohomologies of symplectic manifolds, viii

symplectic cohomologies, v

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- $H^{\bullet}_{BC,\omega}$ contains informations on the symplectic struct, and
- $H^{\bullet}_{BC,\omega} \to H^{\bullet}_{dR}$ allows their comparison with top aspects.

Cohomologies of symplectic manifolds, ix
cohomological decompositions, i

The surjectivity of the map

$$H^{\bullet}_{BC,\omega}(X) \to H^{\bullet}_{dR}(X;\mathbb{R})$$

yields to a natural symplectic decomposition of de Rham cohom.

J.-L. Brylinski, A differential complex for Poisson manifolds, *J. Differ. Geom.* 28 (1988), no. 1, 93–114.

Cohomologies of symplectic manifolds, x
cohomological decompositions, ii

The surjectivity of the map

$$H^{\bullet}_{BC,\omega}(X) \to H^{\bullet}_{dR}(X;\mathbb{R})$$

yields to a natural symplectic decomposition of de Rham cohom.

It corresponds to: each de Rham class admits a d-closed d^{Λ}-closed representative:

Conjecture 2.2.7. If M is a symplectic manifold which is compact, any cohomology class in $H^*(M, \mathbf{C})$ has a representative α such that $d\alpha = \delta\alpha = 0$.

J.-L. Brylinski, A differential complex for Poisson manifolds, *J. Differ. Geom.* 28 (1988), no. 1, 93–114.

Cohomologies of symplectic manifolds, xi

cohomological decompositions, iii

Thm (Mathieu, Yan, Merkulov, Guillemin, Cavalcanti)

Let X be a compact $2n$-mfd endowed with ω symplectic. The following are equivalent:

- *Brylinski's conj: any de Rham class has d-closed d^Λ-closed repres;*
- *Brylinski's $\mathcal{C}^\infty$-fullness: $H^\bullet_{BC,\omega} \to H^\bullet_{dR}$ surj;*
- *Hard Lefschetz Condition: $[\omega^k] \smile \cdot : H^{n-k}_{dR} \to H^{n+k}_{dR}$ isom, $\forall k$;*
- *$\mathrm{d}\,\mathrm{d}^\Lambda$-Lemma: $H^\bullet_{BC,\omega} \to H^\bullet_{dR}$ inj;*
- *sympl cohom relation: $H^\bullet_{BC,\omega} \to H^\bullet_{dR}$ isom;*
- *Lefschetz dec in cohom: $H^\bullet_{dR} = \bigoplus_k L^k PH^{\bullet-2k}$.*

O. Mathieu, Harmonic cohomology classes of symplectic manifolds, *Comment. Math. Helv.* 70 (1995), no. 1, 1–9.

D. Yan, Hodge structure on symplectic manifolds, *Adv. Math.* 120 (1996), no. 1, 143–154.

G. R. Cavalcanti, New aspects of the dd^c-lemma, Oxford University D. Phil thesis, arXiv:math/0501406v1 [math.DG].

Cohomologies of symplectic manifolds, xii

cohomological decompositions, iv

A weaker symplectic decomposition property:

- Lefschetz decomposition moves to de Rham cohom:

$$H^\bullet_{dR} = \bigoplus_{r,s} H^{(r,s)}_\omega(X)$$

where

$$H^{(r,s)}_\omega(X) := \left\{[\alpha] \in H^{2r+s}_{dR}(X;\mathbb{R}) \;:\; \alpha \in L^r P^s\right\} .$$

—, A. Tomassini, Symplectic manifolds and cohomological decomposition, *J. Symplectic Geom.* 12 (2014), no. 2, 215–236.

Cohomologies of symplectic manifolds, xiii
cohomological decompositions, v

Thm (—, A. Tomassini)

Let X be a cpt $2n$-mfd endowed with ω symplectic. Then

$$H^2_{dR}(X;\mathbb{R}) \;=\; H^{(1,0)}_{\omega}(X) \oplus H^{(0,2)}_{\omega}(X)\,.$$

Ch. Benson, C. S. Gordon, Kähler and symplectic structures on nilmanifolds, *Topology* 27 (1988), no. 4, 513–518.

M. Rinaldi, Proprietà coomologiche di varietà simplettiche, Tesi di Laurea, Università degli Studi di Parma, a.a. 2012/2013.

Cohomologies of symplectic manifolds, xiv
cohomological decompositions, vi

Thm (—, A. Tomassini)

Let X be a cpt $2n$-mfd endowed with ω symplectic. Then

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In part, if $2n = 4$, then Lefschetz dec moves to de Rham cohom.

Ch. Benson, C. S. Gordon, Kähler and symplectic structures on nilmanifolds, *Topology* 27 (1988), no. 4, 513–518.

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Cohomologies of symplectic manifolds, xv
cohomological decompositions, vii

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But non-tori nilmanifolds does not satisfy HLC (Benson and Gordon).

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Cohomologies of symplectic manifolds, xvi
cohomological decompositions, viii

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But non-tori nilmanifolds does not satisfy HLC (Benson and Gordon).

Higher-dim non-HLC mfds for which Lefschetz dec moves to de Rham cohom by Rinaldi.

Ch. Benson, C. S. Gordon, Kähler and symplectic structures on nilmanifolds, *Topology* 27 (1988), no. 4, 513–518.

M. Rinaldi, Proprietà coomologiche di varietà simplettiche, Tesi di Laurea, Università degli Studi di Parma, a.a. 2012/2013.

Cohomologies of symplectic manifolds, xvii

inequality *à la* Frölicher for symplectic structures, i

Thm (—, A. Tomassini)

Let X be a $2n$-dim cpt symplectic mfd.

Cohomologies of symplectic manifolds, xviii

inequality *à la* Frölicher for symplectic structures, ii

Thm (—, A. Tomassini)

Let X be a $2n$-dim cpt symplectic mfd. Then, for any k,

$$\dim_{\mathbb{R}} H^k_{BC,\omega}(X) + \dim_{\mathbb{R}} H^k_{A,\omega} \ \geq\ 2 \dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R}) \ .$$

Cohomologies of symplectic manifolds, xix

inequality *à la* Frölicher for symplectic structures, iii

Thm (—, A. Tomassini)

Let X be a $2n$-dim cpt symplectic mfd. Then, for any k,

$$\dim_{\mathbb{R}} H^k_{BC,\omega}(X) + \dim_{\mathbb{R}} H^k_{A,\omega} \geq 2 \dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R}) .$$

Furthermore, the equality

$$\dim_{\mathbb{R}} H^k_{BC,\omega}(X) + \dim_{\mathbb{R}} H^k_{A,\omega} = 2 \dim_{\mathbb{R}} H^k_{dR}(X;\mathbb{R})$$

holds for any k if and only if X satisfies $d\, d^{\Lambda}$-Lemma.

—, A. Tomassini, Inequalities *à la* Frölicher and cohomological decomposition, to appear in *J. Noncommut. Geom.*.

Cohomologies of symplectic manifolds, xx

inequality *à la* Frölicher for symplectic structures, iv

(Generalized-)complex case:

- for compact complex manifolds: (—, A. Tomassini)

$$\begin{aligned} \dim_{\mathbb{C}} H^{\bullet}_{BC}(X) + \dim_{\mathbb{C}} H^{\bullet}_{A} &\geq \dim_{\mathbb{C}} H^{\bullet}_{\partial}(X) + \dim_{\mathbb{C}} H^{\bullet}_{\bar\partial}(X) \\ &\geq 2 \dim_{\mathbb{C}} H^{\bullet}_{dR}(X;\mathbb{C}) \end{aligned}$$

  and equalities hold *iff* $\partial\bar\partial$-Lemma;
- the results can be generalized to generalized-complex structures. (—, A. Tomassini; Chan, Suen)

—, A. Tomassini, On the $\partial\bar\partial$-Lemma and Bott-Chern cohomology, *Invent. Math.* 192 (2013), no. 1, 71–81.

K. Chan, Y.-H. Suen, A Frölicher-type inequality for generalized complex manifolds, arXiv:1403.1682 [math.DG].

The point, in the symplectic case, is that the "associated" Frölicher spectral sequences degenerate at the first step.

J.-L. Brylinski, A differential complex for Poisson manifolds, *J. Differ. Geom.* 28 (1988), no. 1, 93–114.

M. Fernández, R. Ibáñez, M. de León, The canonical spectral sequences for Poisson manifolds, *Isr. J. Math.* 106 (1998), no. 1, 133–155.

$X = \Gamma\backslash G$ nilmanifold (compact quotients of connected simply-connected nilpotent Lie groups G by co-compact discrete subgroups Γ).

Cohomologies of symplectic manifolds, xxiii
symplectic cohomologies of nil/solv-manifolds, ii

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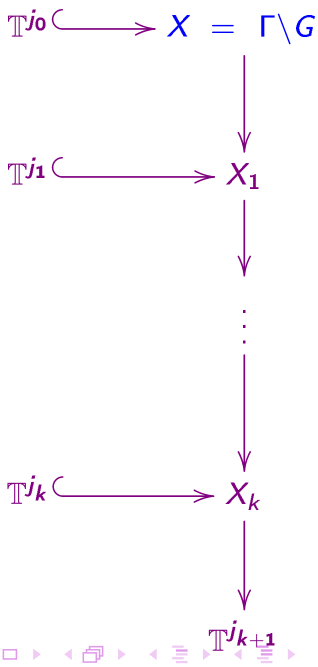


Cohomologies of symplectic manifolds, xxiv
symplectic cohomologies of nil/solv-manifolds, iii

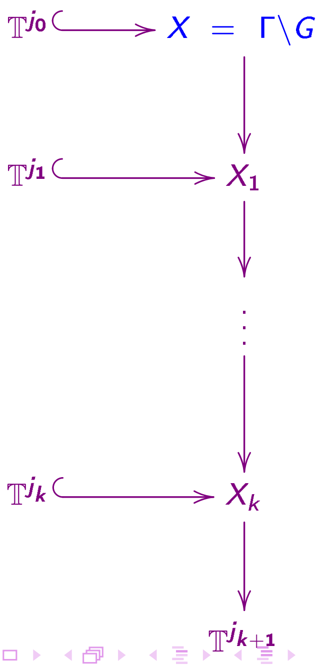
$X = \Gamma \backslash G$ nilmanifold (compact quotients of connected simply-connected nilpotent Lie groups G by co-compact discrete subgroups Γ).

Consider the finite-dim space of forms invariant for left-action $G \curvearrowright X$:

$$\iota \colon \wedge^\bullet \mathfrak{g}^* \hookrightarrow \wedge^\bullet X \ .$$

Thm (Nomizu)

For nilmanifolds, $H_{dR}(\iota)$ isom.



Cohomologies of symplectic manifolds, xxv
symplectic cohomologies of nil/solv-manifolds, iv

$X = \Gamma \backslash G$ solvmanifold (compact quotients of connected simply-connected solvable Lie groups G by co-compact discrete subgroups Γ).

P. de Bartolomeis, A. Tomassini, On solvable generalized Calabi-Yau manifolds, *Ann. Inst. Fourier (Grenoble)* 56 (2006), no. 5, 1281–1296.

H. Kasuya, Minimal models, formality and hard Lefschetz properties of solvmanifolds with local systems, *J. Differ. Geom.* 93 (2013), 269–298.

Cohomologies of symplectic manifolds, xxvi
symplectic cohomologies of nil/solv-manifolds, v

$X = \Gamma \backslash G$ solvmanifold (compact quotients of connected simply-connected solvable Lie groups G by co-compact discrete subgroups Γ).

In this case, de Rham cohomology may depend on the lattice.

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Cohomologies of symplectic manifolds, xxvii

symplectic cohomologies of nil/solv-manifolds, vi

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In this case, de Rham cohomology may depend on the lattice.

Thm (Kasuya)

For solvmanifolds $X = \Gamma \backslash G$, there exists a finite-dim sub-algebra

$$\iota\colon\ (A^\bullet, \mathrm{d}) \hookrightarrow (\wedge^\bullet X, \mathrm{d})$$

such that $H_{dR}(\iota)$ isom.

P. de Bartolomeis, A. Tomassini, On solvable generalized Calabi-Yau manifolds, *Ann. Inst. Fourier (Grenoble)* 56 (2006), no. 5, 1281–1296.

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Cohomologies of symplectic manifolds, xxviii

symplectic cohomologies of nil/solv-manifolds, vii

Take $X = \Gamma / G$ nilmfd (*resp.*, solvmfd).

M. Macrì, Cohomological properties of unimodular six dimensional solvable Lie algebras, arXiv:1111.5958v2 [math.DG].

—, H. Kasuya, Symplectic Bott-Chern cohomology of solvmanifolds, arXiv:1308.4258 [math.SG].

Cohomologies of symplectic manifolds, xxix
symplectic cohomologies of nil/solv-manifolds, viii

Take $X = \Gamma / G$ nilmfd (*resp.*, solvmfd).

Suppose $\omega \in \wedge^2 \mathfrak{g}^*$ (*resp.*, $\omega \in A^2_\Gamma \cap \wedge^2 \mathfrak{g}^*$) symplectic.

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[iNδAM]

Cohomologies of symplectic manifolds, xxx
symplectic cohomologies of nil/solv-manifolds, ix

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Also d^Λ-cohomology can be computed by just $\wedge^\bullet \mathfrak{g}^*$ (*resp.*, $A^\bullet_\Gamma$).

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[iNδAM]

Cohomologies of symplectic manifolds, xxxi

symplectic cohomologies of nil/solv-manifolds, x

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[iNδAM]

Cohomologies of symplectic manifolds, xxxii

symplectic cohomologies of nil/solv-manifolds, xi

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Also d^Λ-cohomology can be computed by just $\wedge^\bullet \mathfrak{g}^*$ (*resp.*, $A^\bullet_\Gamma$). In fact, it is isom to de Rham cohom via Brylinski-$\star$-operator.

Thm (—, H. Kasuya)

Let $X = \Gamma / G$ solvmfd with $\omega \in A^2_\Gamma$ left-inv symplectic. Then symplectic cohomologies $H^\bullet_{BC,\omega}$ and $H^\bullet_{A,\omega}$ are computed by $(A^\bullet_\Gamma, d)$.

M. Macrì, Cohomological properties of unimodular six dimensional solvable Lie algebras, arXiv:1111.5958v2 [math.DG].

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[iNδAM]

Generalized-complex structures:

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- cplx structure:
  $J\colon TX \stackrel{\simeq}{\to} TX$ satisfying an algebraic condition ($J^2 = -\mathrm{id}_{TX}$) and an analytic condition (integrability in order to have holomorphic coordinates).

Generalized-complex geometry, iii
generalized-complex structures, iii

Generalized-complex structures:

- cplx structure:
  $J\colon TX \xrightarrow{\simeq} TX$ satisfying an algebraic condition ($J^2 = -\operatorname{id}_{TX}$) and an analytic condition (integrability in order to have holomorphic coordinates).
- sympl structure:
  $\omega\colon TX \xrightarrow{\simeq} T^*X$ satisfying an algebraic condition (ω non-deg 2-form) and an analytic condition ($\mathrm{d}\,\omega = 0$).

[iNδAM]

Generalized-complex geometry, iv
generalized-complex structures, iv

Hence, consider the bundle $TX \oplus T^*X$.

N. J. Hitchin, Generalized Calabi-Yau manifolds, *Q. J. Math.* 54 (2003), no. 3, 281–308.

M. Gualtieri, Generalized complex geometry, Oxford University DPhil thesis, arXiv:math/0401221 [math.DG].

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[iNδAM]

Generalized-complex geometry, v

generalized-complex structures, v

Hence, consider the bundle $TX \oplus T^*X$. Note that it admits a natural bilinear pairing: $\langle X + \xi | Y + \eta \rangle = \frac{1}{2} \left(\iota_X \eta + \iota_Y \xi \right)$.

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Mimicking the def of cplx and sympl structures:

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G. R. Cavalcanti, New aspects of the dd^c-lemma, Oxford University D. Phil thesis, arXiv:math/0501406 [math.DG].

Hence, consider the bundle $TX \oplus T^*X$. Note that it admits a natural bilinear pairing: $\langle X+\xi | Y+\eta \rangle = \frac{1}{2}\left(\iota_X \eta + \iota_Y \xi\right)$.

Mimicking the def of cplx and sympl structures:

a generalized-complex structure on a $2n$-dim mfd X is a

$$\mathcal{J} \colon TX \oplus T^*X \to TX \oplus T^*X$$

such that $\mathcal{J}^2 = -\operatorname{id}_{TX\oplus T^*X}$, being orthogonal wrt $\langle - | = \rangle$, and satisfying an integrability condition.

N. J. Hitchin, Generalized Calabi-Yau manifolds, *Q. J. Math.* 54 (2003), no. 3, 281–308.

M. Gualtieri, Generalized complex geometry, Oxford University DPhil thesis, arXiv:math/0401221 [math.DG].

G. R. Cavalcanti, New aspects of the dd^c-lemma, Oxford University D. Phil thesis, arXiv:math/0501406 [math.DG].

Generalized-cplx geom unifies cplx geom and sympl geom:

Generalized-complex geometry, ix
generalized-complex structures, ix

Generalized-cplx geom unifies cplx geom and sympl geom:

- J cplx struct:

$$\mathcal{J} = \left(\begin{array}{c|c} -J & 0 \\ \hline 0 & J^* \end{array}\right) \qquad \text{gen-cplx;}$$

- ω sympl struct:

$$\mathcal{J} = \left(\begin{array}{c|c} 0 & -\omega^{-1} \\ \hline \omega & 0 \end{array}\right) \qquad \text{gen-cplx.}$$

Generalized-complex geometry, x
generalized-complex cohomologies, i

As in cplx/sympl cases, generalized-complex structures gives

$$\left(U^\bullet, \partial, \overline{\partial}\right) \quad \text{bi-diff graded algebra} .$$

Generalized-complex geometry, xi
generalized-complex cohomologies, ii

As in cplx/sympl cases, generalized-complex structures gives

$$\left(U^\bullet,\, \partial,\, \overline{\partial}\right) \quad \text{bi-diff graded algebra}\ .$$

Hence $\rightsquigarrow$ generalized-cplx cohomologies: $GH^\bullet_{\overline{\partial}}$, $GH^\bullet_{BC}$, $GH^\bullet_{A}$.

Generalized-complex geometry, xii
generalized-complex cohomologies, iii

Explicit examples can be obtained by:

Thm (—, S. Calamai)

For nilmanifolds with "suitable" gen-cplx structures, generalized-complex cohomologies are computed by just left-invariant forms.

—, S. Calamai, Cohomologies of generalized-complex manifolds and nilmanifolds, arXiv:1405.0981 [math.DG].

Generalized-complex geometry, xiii
generalized-complex cohomologies, iv

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Generalized-complex geometry, xiv
generalized-complex cohomologies, v

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—, S. Calamai, Cohomologies of generalized-complex manifolds and nilmanifolds, `arXiv:1405.0981 [math.DG]`.

Generalized-complex geometry, xv
generalized-complex cohomologies, vi

When

$$GH^{\bullet}_{BC}(X) \to GH_{dR}(X) \quad \text{surj},$$

we have gen-cplx cohomological decomposition of de Rham.

—, S. Calamai, A. Latorre, On cohomological decomposition of generalized-complex structures, arXiv:1406.2101 [math.DG].

T.-J. Li, W. Zhang, Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds, *Comm. Anal. Geom.* 17 (2009), no. 4, 651–684.

Generalized-complex geometry, xvi
generalized-complex cohomologies, vii

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[iNδAM]

Generalized-complex geometry, xvii
generalized-complex cohomologies, viii

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- In the sympl case, it coincides with Brylinski's $\mathcal{C}^\infty$-fullness, equiv, $d\, d^\Lambda$-Lemma.

—, S. Calamai, A. Latorre, On cohomological decomposition of generalized-complex structures, arXiv:1406.2101 [math.DG].

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Generalized-complex geometry, xviii
generalized-complex cohomologies, ix

Example:

moduli-space of left-inv cplx structures on Iwasawa manifold has two connected components.

G. R. Cavalcanti, M. Gualtieri, Generalized complex structures on nilmanifolds, *J. Symplectic Geom.* 2 (2004), no. 3, 393–410.

Generalized-complex geometry, xix
generalized-complex cohomologies, x

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- Abelian and holomorphically-parallelizable cplx structures are $\mathcal{C}^\infty$-pure-and-full in sense of Li and Zhang.

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Generalized-complex geometry, xx
generalized-complex cohomologies, xi

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G. R. Cavalcanti, M. Gualtieri, Generalized complex structures on nilmanifolds, *J. Symplectic Geom.* 2 (2004), no. 3, 393–410.

Generalized-complex geometry, xxi

generalized-complex cohomologies, xii

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  ...but connected as gen-cplx structures by Cavalcanti and Gualtieri.

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Generalized-complex geometry, xxii

generalized-complex cohomologies, xiii

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- In fact, the connecting gen-cplx structure is $\mathcal{C}^\infty$-full as gen-cplx (—, S. Calamai, A. Latorre).

G. R. Cavalcanti, M. Gualtieri, Generalized complex structures on nilmanifolds, *J. Symplectic Geom.* 2 (2004), no. 3, 393–410.

GeCo GeDi project:

http://gecogedi.dimai.unifi.it

Joint work with: Adriano Tomassini, Hisashi Kasuya, Simone Calamai, Adela Latorre, Federico A. Rossi, Maria Giovanna Franzini, Weiyi Zhang, Georges Dloussky.

And with the fundamental contribution of: Serena, Alessandra, Maria Beatrice and Luca, Maria Rosaria, Francesco, Anna Rita, Andrea, Chiara, Michele, Laura, . . .