YAU'S CONJECTURE FOR THE STACKED CLIFFORD TORI OF WIYGUL

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Abstract. We prove Yau's conjecture $\lambda_1 = 2$ for the stacked Clifford tori of Wiygul: for all integers $N \geq 2$, $k, \ell \geq 1$ and every sufficiently large m, the closed embedded minimal surface of genus $k\ell m^2(N-1)+1$ in the round three-sphere which resembles N parallel copies of the Clifford torus joined by small catenoidal tunnels has first Laplace eigenvalue 2. For $N \geq 3$ these surfaces are chains rather than doublings, and the even–odd decomposition on which all previous verifications for gluing constructions rest is not available. The reflection lemma of Choe and Soret reduces the problem to the sector of functions invariant under the symmetry group of the construction, and we show that the lowest nonzero eigenvalue of that sector equals $4 + O(m^{-1})$. The value 4 is the outcome of an exact identity: the limiting waist ratios of the construction form the Perron vector of the adjacency operator of the line graph of a path, so that the spectral gap of the path cancels against the total conductance of the tunnels prescribed by the balancing conditions, and what survives is the coefficient of the Jacobi operator of the Clifford torus. The analytic input consists of a conformally invariant channel inequality on a cylinder and of a Poincaré inequality on a periodically perforated torus. The properties of the construction on which the argument rests are isolated in a reduction theorem for closed surfaces decomposed into blocks joined by families of thin channels along the edges of a finite graph, subject to a symmetry assumption and to Poincaré and trace inequalities on the blocks.

Contents

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1. Introduction

1.1. **Yau's conjecture.** Let Σ^n be a closed minimal hypersurface of the round unit sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ and let $0 = \lambda_0(\Sigma) < \lambda_1(\Sigma) \leq \lambda_2(\Sigma) \leq \cdots$ be the spectrum of its Laplace–Beltrami operator. By a theorem of Takahashi [20] the restrictions to Σ of the coordinate functions $x_1, \ldots, x_{n+2}$ of $\mathbb{R}^{n+2}$ satisfy

$$\Delta_\Sigma x_i + n\, x_i = 0, \tag{1.1}$$

so that each $x_i|_\Sigma$ has vanishing mean and is admissible in the variational characterisation of λ_1; hence $\lambda_1(\Sigma) \leq n$. In his problem list Yau [24, Problem 100] asked whether equality holds for embedded hypersurfaces.

Yau's Conjecture. If Σ^n is a closed embedded minimal hypersurface of $\mathbb{S}^{n+1}$, then $\lambda_1(\Sigma) = n$.

To our knowledge the conjecture is open in every dimension. An affirmative answer implies genus-dependent area bounds for embedded minimal surfaces of $\mathbb{S}^3$ [23] and, through the theorem of Montiel and Ros on minimal immersions by first eigenfunctions [18], Lawson's conjecture that the Clifford torus is the only embedded minimal torus of $\mathbb{S}^3$, proved by Brendle [1]. Two kinds of partial results are known. Choi and Wang [4] proved the universal bound $\lambda_1 \geq n/2$ by Reilly's formula on the two components of $\mathbb{S}^{n+1} \setminus \Sigma$, and quantitative refinements were obtained in [6, 7]; all of these remain strictly below n. In the other direction, equality has been established on explicit families: Tang and Yan [21] for all minimal isoparametric hypersurfaces, Choe and Soret [3] for the surfaces of Lawson [17] and of Karcher, Pinkall and Sterling [14], and Kapouleas and McGrath [10] for the doublings of the equatorial two-sphere constructible by the linearised doubling theorem of [9], which develops [8], and for the doublings of the Clifford torus constructed in [10].

1.2. **Doublings and chains.** All verifications for gluing constructions concern *doublings*: two nearly parallel copies of a base surface joined by small catenoidal tunnels. There, exactly or after a small perturbation of the metric, an involutive isometry exchanges the two copies, and each eigenspace splits into an even part, controlled by comparison with the base surface, and an odd part, controlled by the tunnels. This dichotomy is the backbone of [3, 10].

For $N \geq 3$ copies arranged in a chain there is no involution whose orbits are the copies and no such dichotomy. The functions which are constant on each copy form an N-dimensional space, and their Rayleigh quotients are governed by the combinatorial Laplacian $L(P_N)$ of the path graph on N vertices, whose spectral gap $2(1-\cos(\pi/N))$ decays like $\pi^2 N^{-2}$. If the tunnels joining consecutive copies had a conductance independent of N, the lowest such quotient would tend to zero and long chains would violate Yau's conjecture. That this does not happen, and why, is the content of this paper.

1.3. **Stacked Clifford tori.** Wiygul [22], extending the doubling construction of Kapouleas and Yang [13], proved that for all integers $N \geq 2$, $k, \ell \geq 1$ and every sufficiently large m there is a closed embedded minimal surface $\Sigma[N, k, \ell, m] \subset \mathbb{S}^3$ of genus $k\ell m^2(N-1)+1$, invariant under a group $\mathcal{G}[k, \ell, m] \cong D_{km} \times D_{\ell m}$ of isometries, which resembles N nested small perturbations of a fixed Clifford torus $\mathbb{T}$ joined by $k\ell m^2$ small catenoidal tunnels between each pair of neighbouring tori, the tunnels of consecutive layers sitting over two interlaced rectangular lattices; as $m \to \infty$ the surfaces converge to $\mathbb{T}$ with multiplicity N. The surface is obtained as a normal graph over an explicit initial surface, and every quantity of the construction which enters the present argument (lattices, gluing radius, waist radii, balancing conditions, conformal lengths of the tunnels) is explicit. In Section 3 we reproduce these data and we isolate the single property of the final surface which is used, namely that its induced metric is uniformly close to that of the initial surface; we verify from the final estimate of [22] that the surfaces constructed there have this property. A closed embedded minimal surface with these features is called a *stacked Clifford torus of type* (N, k, ℓ, m) (Definition 3.5); all the surfaces of [22] are of this kind (Corollary 3.8).

1.4. **Results.** Throughout, $\mathcal{G} = \mathcal{G}[k, \ell, m]$ is the symmetry group of the construction, and the $\mathcal{G}$-invariant sector of the Laplacian of Σ is its restriction to $\mathcal{G}$-invariant functions.

Theorem A. Let $N \geq 2$ and $k, \ell \geq 1$ be integers. There exists $m_0 = m_0[N, k, \ell]$ such that for every $m \geq m_0$ every stacked Clifford torus Σ of type (N, k, ℓ, m) satisfies

$$\lambda_1(\Sigma) = 2,$$

and the smallest nonzero eigenvalue of the $\mathcal{G}$-invariant sector of the Laplacian of Σ equals $4 + O(m^{-1})$, the constant in the error term depending on N, k and ℓ. In particular, after enlarging m_0 if necessary, Yau's conjecture holds for every surface constructed in [22] with $m \geq m_0$.

The range $N \geq 3$ is the substance of Theorem A. For $N = 2$ the surfaces are doublings; the Clifford torus doublings of [10] are treated there by the even–odd method, and we make no claim of priority in the case $N = 2$.

The threshold m_0 is neither effective, since it must exceed the non-explicit threshold of [22, Theorem 6.49], nor uniform in N. The conformal half-length a_i of the tunnels is $k\ell\pi m^2/(8N^2)$ to leading order (Proposition 5.4 and Remark 5.5), so that the hypothesis $m \geq m_0$, already in [22], entails $m \gg N$, and the relative errors in Theorem B below are of size $N^2/(k\ell m^2)$. What is uniform in N is the leading value 4, and this is exactly what excludes the collapse of the spectral gap described in Section 1.2.

The linearised balancing conditions of the construction determine the ratios $b_i = \tau_i/\tau_1$ of the waist radii of the tunnels in the successive layers, together with a constant b_2 which enters the exponent of the waist radii; as $m \to \infty$ they converge to limiting values $b_i[N]$ determined by a tridiagonal system (Section 3). Write $L(P_N)$ and $A(P_N)$ for the combinatorial Laplacian and the adjacency operator of the path graph on N vertices, D for an oriented incidence matrix of P_N, and $\lambda_1(L(P_N))$ for the spectral gap.

Theorem B. Let $N \geq 2$.

(i) The limiting waist ratios are

$$b_i[N] = \frac{\sin(i\pi/N)}{\sin(\pi/N)}, \qquad 1 \leq i \leq N-1,$$

so that the balancing constant is $b_2[N] = 2\cos(\pi/N)$, the largest eigenvalue of $A(P_{N-1})$, and $\lambda_1(L(P_N)) = 2 - b_2[N]$.

(ii) For every stacked Clifford torus of type (N, k, ℓ, m) the total conductance of the tunnels joining two adjacent tori is

$$C_i = \frac{8\pi^2}{2 - b_2[N]}\big(1 + O(m^{-2})\big) =: C_N\big(1 + O(m^{-2})\big), \qquad \frac{C_N}{2\pi^2}\,\lambda_1\big(L(P_N)\big) = 4,$$

and the lowest Rayleigh quotient of the weighted path graph model of the invariant sector (Definition 6.7) equals $4 + O(m^{-2})$; the constants in the error terms depend on N, k and ℓ.

(iii) As $N \to \infty$, $\lambda_1(L(P_N)) = \pi^2 N^{-2} + O(N^{-4})$ and $C_N = 8N^2 + O(1)$.

The cancellation in (ii) is exact and uses no closed form: it rests on the incidence identity $L(P_N) = DD^{\mathsf{T}}$, $D^{\mathsf{T}}D = 2I - A(P_{N-1})$, which identifies the nonzero spectrum of $L(P_N)$ with $2 - \operatorname{spec} A(P_{N-1})$, and on the observation that the vector of limiting waist ratios is a *positive* eigenvector of $A(P_{N-1})$ with eigenvalue $b_2[N]$, hence the Perron vector. The constant 4 that survives is the coefficient of the Jacobi operator

$$\mathcal{J}_{\mathbb{T}} = \Delta_{\mathbb{T}} + |A|^2 + \operatorname{Ric}(\nu,\nu) = \Delta_{\mathbb{T}} + 4 \tag{1.2}$$

of the Clifford torus, which enters the construction through the linearisation producing the balancing conditions. In Section 8 we prove that among trees the identity $\lambda_1(L(T)) = 2 - \lambda_{\max}(A(\operatorname{Line}(T)))$ characterises paths (Proposition 8.2); the mechanism is therefore intrinsically one-dimensional.

1.5. **An abstract reduction.** The proof of Theorem A uses only a short list of properties of the surfaces Σ, and it is convenient to isolate them. Suppose a closed surface is cut into *blocks* indexed by the vertices of a finite connected graph $\mathsf{G} = (V, E)$ and *channels* indexed by its edges, each edge e carrying M_e channels conformally modelled on a flat cylinder $[-a_e, a_e] \times \mathbb{S}^1$ whose conformal factor decays exponentially away from the ends, where it is of size r_e; suppose a finite group $\mathcal{G}$ of isometries preserves each block, permutes the channels of each edge and acts on their coordinates by $(t,\theta) \mapsto (t, \pm\theta + \theta_0)$; and suppose a Poincaré inequality with constant P and an aggregate trace inequality with constant K hold on each block for $\mathcal{G}$-invariant functions. This is Definition 6.9. Let A_v be the area of the v-th block, $\Theta_v := \sum_{e \ni v} M_e \pi r_e^2$ the area which the channels have removed from it, $\widetilde{A}_v := A_v + \Theta_v$, and $C_e := M_e\pi/a_e$ the total conductance of the e-th family; let $\mu(\widetilde{A}, C)$ be the lowest Rayleigh quotient of the graph G with these vertex and edge weights (Definition 6.7), and set

$$\Xi := \varepsilon + \eta + \mathsf{P} + \sum_{e \in E} M_e^{1/2} r_e^2 + \sum_{e \in E} M_e^{-1/2},$$

where ε measures the conformal distortion of the channels and η the deviation of the area of a half-channel from πr_e^2.

Theorem C. There are constants $C^* > 0$ and $\Xi_0 > 0$, depending only on $|V|$, $|E|$, on two-sided bounds for the $\widetilde{A}_v$, on an upper bound for the C_e and on K, such that if $\Xi \le \Xi_0$ then

$$\Big| \min\Big\{\mathcal{R}(u) : u \in H^1(\Sigma)^{\mathcal{G}} \setminus \{0\},\ \textstyle\int_\Sigma u = 0\Big\} - \mu(\widetilde{A}, C) \Big| \le C^* \,\Xi.$$

If in addition $\Sigma \subset \mathbb{S}^3$ is a closed embedded minimal surface, $\mathcal{G} < O(4)$ is generated by reflections in great spheres, and $\mu(\widetilde{A}, C) > 2 + C^*\Xi$, then $\lambda_1(\Sigma) = 2$.

Theorem A is the case where G is the path P_N, the blocks are the tori, the channels are the tunnels, and $\Xi = O(m^{-1})$; the verification is Proposition 6.12. We stress the limits of this statement. Theorem C is a reduction, not an existence result: it converts the spectral problem into the computation of $\mu(\widetilde{A}, C)$, and it says nothing about which weights occur. Both inputs which make the computation succeed here, namely the exponential decay of the conformal factor along a catenoidal neck and the balancing conditions which fix the r_e, must be supplied by the construction at hand, and verifying the hypotheses for another family requires work of the kind carried out in Sections 3 and 6. Doublings and stackings of the equatorial two-sphere, and stackings of the equatorial disc in the free boundary setting, seem natural candidates; we make no claim about them here.

1.6. **Method.** By Takahashi's theorem $\lambda_1(\Sigma) \le 2$. If $\lambda_1(\Sigma) < 2$, the reflection lemma of Choe and Soret [3], in a formulation which uses only minimality, Takahashi's theorem and Courant's nodal domain theorem (Lemma 2.2), shows that every first eigenfunction is invariant under the group $\mathcal{G}$, which is generated by reflections in great spheres. It therefore suffices to bound the Rayleigh quotient from below on the $\mathcal{G}$-invariant sector. On each torus the $\mathcal{G}$-invariant functions have Fourier support in a lattice of frequencies of order m, so a $\mathcal{G}$-invariant function is, up to an error controlled by its Dirichlet energy, constant on each torus; the Dirichlet energy of such a function is bounded from below by the energy of the interpolating mode in the tunnels, and the latter is controlled by a sharp channel inequality. The resulting lower bound is the lowest Rayleigh quotient μ of a weighted path graph whose weights are the areas of the tori and the conductances of the tunnel families, and Theorem B evaluates $\mu = 4 + O(m^{-2})$. Two elementary inequalities carry the analysis: the channel inequality on a cylinder (Lemma 5.1), which is conformally invariant and imposes no condition on the boundary data, so that it applies to the restriction of an arbitrary function to a tunnel; and a Poincaré inequality on the periodically perforated torus (Lemma 6.3), available with a constant independent of m because the ratio of the gluing radius to the lattice spacing is fixed in [22]. We use neither capacity theory nor spectral convergence on graph-like spaces, and we work on the minimal surface itself.

1.7. **What is used from the construction.** Numbered references to [22] follow the arXiv version arXiv:1502.07420v2. From that paper we use: the definition of the initial surfaces [22, (2.1)–(2.9), (2.26)–(2.30)]; the definition of the waist radii and the balancing conditions [22, (2.12)–(2.18), (3.11)–(3.17)] together with the existence statement [22, Lemma 3.18] and the bounds [22, (3.6)] on the heights; the parametrisation [22, (4.20)–(4.23)] of the catenoidal regions; the statement [22, (4.4)] that $\mathcal{G}$ acts on normal perturbations without sign; the conformal factor ρ of [22, (4.9)–(4.13), (4.25)], the weighted Hölder norms of [22, (4.6)] and the decay norms of the paragraph "Decay norms and a global estimate of the mean curvature" of [22, §4], whose exponents $\alpha, \gamma \in (0, 1)$ are fixed once and for all in the hypotheses of [22, Theorem 6.49]; and the main theorem [22, Theorem 1.1, Lemma 4.40, Theorem 6.49, (6.60)], which produces the minimal surface as the normal graph over the initial surface of a function u satisfying $\|u\|_{2,\alpha,\gamma} \le 2C_1\tau_1$. In Section 3.7 we show that this estimate implies property (W) of Definition 3.5, the only property of the final surface used afterwards; the derivation uses of the weight only the positivity of its exponent γ, and of ρ only the two-sided bound $m \le \rho \le C\tau_1^{-1}$. All estimates on the initial surface which we need are derived in Section 3 from the explicit formulas.

1.8. **Organisation.** Section 2 collects Takahashi's theorem and the reflection lemma. Section 3 records the data of the construction, defines the class of surfaces and shows that the surfaces of [22] belong to it. Section 4 proves Theorem B(i). Section 5 proves the channel inequality and evaluates the conductances. Section 6 establishes the Poincaré, trace and mass estimates, proves the abstract reduction theorem (Theorem C, restated as Theorem 6.11) and applies it to the invariant sector. Section 7 proves Theorems A and B. Section 8 discusses the structure of the identity, analyses the balancing operator of a tree, characterises paths among trees, and states two problems.

Convention 1.1. All surfaces are smooth, closed, embedded and connected. The Laplacian is $\Delta = \operatorname{div}\nabla$, so that $\Delta\varphi + \lambda\varphi = 0$ for an eigenfunction with eigenvalue $\lambda \ge 0$. We write $|\Omega|$ for the area of a two-dimensional region, and

$$\mathcal{R}(u) = \frac{\int_\Sigma |\nabla u|^2}{\int_\Sigma u^2}, \qquad \lambda_1(\Sigma) = \min\Big\{\mathcal{R}(u) : u \in H^1(\Sigma)\setminus\{0\},\ \textstyle\int_\Sigma u = 0\Big\}.$$

Unless stated otherwise, constants denoted C, C', c, K, … depend only on N, k and ℓ and may change from line to line; $O(m^{-1})$ and $O(m^{-2})$ are understood with such constants, uniformly for $m \ge m_0[N,k,\ell]$, and m_0 may be enlarged finitely many times. No uniformity in N is claimed for the constants or for m_0; see Remark 5.5.

2. Preliminaries

Lemma 2.1 (Takahashi [20])**.** *Let $\Sigma \subset \mathbb{S}^3$ be a closed minimal surface. For every linear function ξ on $\mathbb{R}^4$ the restriction $\xi|_\Sigma$ satisfies $\Delta_\Sigma\xi + 2\xi = 0$ and $\int_\Sigma \xi = 0$. Consequently $\lambda_1(\Sigma) \le 2$.*

Proof. The equation is (1.1) with $n = 2$; integrating it over the closed surface gives $\int_\Sigma \xi = 0$. Since $\xi|_\Sigma \not\equiv 0$ for a suitable ξ, the variational characterisation gives $\lambda_1 \le 2$. □

The following lemma is the first half of the argument of Choe and Soret [3]. We include a proof because its hypotheses matter: it uses minimality only through Takahashi's theorem, together with Courant's nodal domain theorem, and nothing else; in particular it is independent of any combinatorial hypothesis on a fundamental domain.

Lemma 2.2 (Reflection lemma)**.** *Let $\Sigma \subset \mathbb{S}^3$ be a closed embedded minimal surface invariant under a finite group $\mathcal{G} < O(4)$ generated by reflections in great spheres. If $\lambda_1(\Sigma) < 2$, then every first eigenfunction of Σ is $\mathcal{G}$-invariant.*

Proof. Assume $\lambda_1 := \lambda_1(\Sigma) < 2$, let u be a real first eigenfunction, let $\sigma \in \mathcal{G}$ be the reflection in a great sphere Π, and set $\psi := u - u\circ\sigma$. Since σ restricts to an isometry of Σ, $u\circ\sigma$ is a λ_1-eigenfunction, so either $\psi \equiv 0$ or ψ is a λ_1-eigenfunction; we assume the latter and derive a contradiction. Note that $\psi\circ\sigma = -\psi$ and that $\psi = 0$ on $\Sigma\cap\Pi$, because σ fixes Π pointwise.

By Courant's nodal domain theorem [5, Ch. VI, §6] a λ_1-eigenfunction has at most two nodal domains. Since $\int_\Sigma \psi = 0$ and $\psi \not\equiv 0$, the open sets $\Omega_+ := \{\psi > 0\}$ and $\Omega_- := \{\psi < 0\}$

are both nonempty; hence each of them is a single nodal domain, and in particular each of them is connected. Since $\psi \circ \sigma = -\psi$, the reflection σ maps Ω_+ onto Ω_-.

Let ξ be a linear function on $\mathbb{R}^4$ vanishing exactly on the hyperplane spanned by Π, so that $\xi \circ \sigma = -\xi$, and let $D_\pm := \Sigma \cap \{\pm\xi > 0\}$. These are disjoint open subsets of Σ with $D_+ \cup D_- = \Sigma \setminus \Pi$, and $\sigma(D_+) = D_-$. Since $\psi = 0$ on $\Sigma \cap \Pi$, the connected set Ω_+ is contained in $\Sigma \setminus \Pi = D_+ \sqcup D_-$, hence in one of the two sets $D_\pm$; replacing u by $-u$, which replaces ψ by $-\psi$ and interchanges Ω_+ and Ω_-, we may assume $\Omega_+ \subset D_+$. Then $\Omega_- = \sigma(\Omega_+) \subset \sigma(D_+) = D_-$. Consequently $\psi \geq 0$ on D_+, because $\{\psi < 0\} = \Omega_- \subset D_-$; likewise $\psi \leq 0$ on D_-; and $\psi = 0$ on $\Sigma \cap \Pi$. Since $\xi > 0$ on D_+, $\xi < 0$ on D_- and $\xi = 0$ on Π, we conclude that $\psi\,\xi \geq 0$ on Σ and $\psi\,\xi > 0$ on the nonempty open set Ω_+; hence $\int_\Sigma \psi\,\xi > 0$.

On the other hand, by Lemma 2.1, $\Delta\xi + 2\xi = 0$ on Σ, while $\Delta\psi + \lambda_1\psi = 0$, so that Green's formula on the closed surface Σ gives

$$(2 - \lambda_1)\int_\Sigma \psi\,\xi = \int_\Sigma \big(\xi\,\Delta\psi - \psi\,\Delta\xi\big) = 0,$$

and $\int_\Sigma \psi\,\xi = 0$ because $\lambda_1 \neq 2$: a contradiction. Therefore $\psi \equiv 0$, that is $u \circ \sigma = u$ for every reflection $\sigma \in \mathcal{G}$ in a great sphere, and u is $\mathcal{G}$-invariant since $\mathcal{G}$ is generated by such reflections. $\square$

Remark 2.3. In [3] the nodal domains of ψ are identified with the two components of $\Sigma \setminus \Pi$ furnished by the two-piece property of Ros [19], and the contradiction is derived on one of them. The orthogonality argument above makes this identification unnecessary: it uses neither the two-piece property nor embeddedness, and it applies verbatim to a closed minimal surface immersed in $\mathbb{S}^3$ on which a reflection of $\mathbb{S}^3$ induces an isometry. We shall use the lemma only for embedded surfaces, which is the setting of Yau's conjecture.

3. The surfaces of Wiygul

In this section all numbered references are to [22], arXiv:1502.07420v2.

3.1. **The Clifford torus and flat coordinates.** Realise $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, let $\mathbb{T} = \{|z_1| = |z_2| = 1/\sqrt{2}\}$ be the Clifford torus, and let $C_1 = \{z_2 = 0\}$ and $C_2 = \{z_1 = 0\}$ be its axes. The map [22, (2.2)]

$$\Phi(\mathrm{x},\mathrm{y},\mathrm{z}) := \Big(e^{i\sqrt{2}\mathrm{x}}\sin\!\big(\mathrm{z} + \tfrac{\pi}{4}\big),\ e^{i\sqrt{2}\mathrm{y}}\cos\!\big(\mathrm{z} + \tfrac{\pi}{4}\big)\Big), \qquad (\mathrm{x},\mathrm{y},\mathrm{z}) \in \mathbb{R}\times\mathbb{R}\times\big(-\tfrac{\pi}{4}, \tfrac{\pi}{4}\big), \tag{3.1}$$

is a covering of $\mathbb{S}^3 \setminus (C_1 \cup C_2)$, and [22, (2.3)]

$$\Phi^* g_{\mathbb{S}^3} = d\mathrm{x}^2 + d\mathrm{y}^2 + d\mathrm{z}^2 + \sin(2\mathrm{z})\big(d\mathrm{x}^2 - d\mathrm{y}^2\big). \tag{3.2}$$

The horizontal planes $\{\mathrm{z} = \text{const}\}$ are mapped onto the tori $\mathbb{T}_\mathrm{z}$ parallel to $\mathbb{T} = \mathbb{T}_0$, on which the induced metric is $(1+\sin 2\mathrm{z})d\mathrm{x}^2 + (1-\sin 2\mathrm{z})d\mathrm{y}^2$ with area element $\cos(2\mathrm{z})\,d\mathrm{x}\,d\mathrm{y}$. In particular $\Phi(\cdot,\cdot,0)$ descends to an isometry of the flat torus

$$\mathbb{T}_{\mathrm{fl}} := \mathbb{R}^2/(\sqrt{2}\pi\mathbb{Z})^2$$

onto $\mathbb{T}$, so that

$$|\mathbb{T}| = 2\pi^2, \tag{3.3}$$

and the eigenvalues of the Laplacian of $\mathbb{T}$ are $2(p^2+q^2)$, $p, q \in \mathbb{Z}$, with eigenfunctions $e^{i\sqrt{2}(p\mathrm{x}+q\mathrm{y})}$. The principal curvatures of $\mathbb{T}$ are ± 1, so $|A|^2 = 2$, and $\mathrm{Ric} = 2g$ on $\mathbb{S}^3$; the Jacobi operator of $\mathbb{T}$ is $\mathcal{J}_\mathbb{T} = \Delta_\mathbb{T} + 4$ as in (1.2). A *parallel torus* is one of the tori $\mathbb{T}_\mathrm{z}$.

3.2. **The symmetry group.** Fix integers $k, \ell, m \geq 1$ and set

$$X := \frac{\pi}{\sqrt{2}\, km}, \qquad Y := \frac{\pi}{\sqrt{2}\, \ell m}, \qquad R := \frac{1}{10\, \ell m}, \qquad M := k\ell m^2. \tag{3.4}$$

Following [22, (2.10)] we assume $k \leq \ell$; this is no loss of generality, since interchanging the axes C_1, C_2 interchanges k and ℓ. Let $\underline{\mathsf{X}}(z_1, z_2) = (\overline{z_1}, z_2)$ and $\underline{\mathsf{Y}}(z_1, z_2) = (z_1, \overline{z_2})$, and let $\mathsf{R}^{\theta}_{C_2}(z_1, z_2) = (e^{i\theta} z_1, z_2)$, $\mathsf{R}^{\theta}_{C_1}(z_1, z_2) = (z_1, e^{i\theta} z_2)$. The symmetry group of the construction is [22, (2.26)]

$$\mathcal{G} = \mathcal{G}[k, \ell, m] := \big\langle \mathsf{R}^{2\pi/km}_{C_2},\ \mathsf{R}^{2\pi/\ell m}_{C_1},\ \underline{\mathsf{X}},\ \underline{\mathsf{Y}} \big\rangle \cong D_{km} \times D_{\ell m}, \tag{3.5}$$

where D_q is the dihedral group of order $2q$. Through Φ [22, (2.25)], $\mathcal{G}$ acts on the coordinates $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ by the group generated by the translations $(\mathrm{x}, \mathrm{y}) \mapsto (\mathrm{x} + 2X, \mathrm{y})$, $(\mathrm{x}, \mathrm{y}) \mapsto (\mathrm{x}, \mathrm{y} + 2Y)$ and the reflections $\mathrm{x} \mapsto -\mathrm{x}$, $\mathrm{y} \mapsto -\mathrm{y}$, the coordinate z being fixed. In particular every element of $\mathcal{G}$ preserves each parallel torus and each of its two sides.

Lemma 3.1. *$\mathcal{G}[k, \ell, m]$ is generated by reflections in great spheres of $\mathbb{S}^3$.*

Proof. Each of $\underline{\mathsf{X}}$, $\underline{\mathsf{Y}}$ fixes pointwise a three-dimensional linear subspace of $\mathbb{R}^4$ and reverses its orthogonal complement, hence is the reflection in a great sphere; so is every conjugate $\mathsf{R}\,\underline{\mathsf{X}}\,\mathsf{R}^{-1}$ by a rotation R about C_1 or C_2. For $\theta \in \mathbb{R}$ one computes $\mathsf{R}^{\theta}_{C_2}\underline{\mathsf{X}}\mathsf{R}^{-\theta}_{C_2}(z_1, z_2) = (e^{2i\theta}\overline{z_1}, z_2)$, that is $\mathsf{R}^{\theta}_{C_2}\underline{\mathsf{X}}\mathsf{R}^{-\theta}_{C_2} = \mathsf{R}^{2\theta}_{C_2}\underline{\mathsf{X}}$. Hence the reflection $\mathsf{R}^{\pi/km}_{C_2}\underline{\mathsf{X}}\mathsf{R}^{-\pi/km}_{C_2} = \mathsf{R}^{2\pi/km}_{C_2}\underline{\mathsf{X}}$ belongs to $\mathcal{G}$, although $\mathsf{R}^{\pi/km}_{C_2}$ itself does not, and together with $\underline{\mathsf{X}}$ it generates the dihedral factor D_{km}, the product $\underline{\mathsf{X}} \cdot \mathsf{R}^{2\pi/km}_{C_2}\underline{\mathsf{X}}$ being $\mathsf{R}^{-2\pi/km}_{C_2}$. Similarly for $D_{\ell m}$. $\square$

3.3. **Lattices, cells and the gluing radius.** Let $\mathsf{L}_0 \subset \mathbb{T}_{\text{fl}}$ be the image of $2X\mathbb{Z} \times 2Y\mathbb{Z}$ and $\mathsf{L}_1 := \mathsf{L}_0 + (X, Y)$ [22, (2.27)]; both are $\mathcal{G}$-invariant rectangular lattices of cell size $2X \times 2Y$, each with $M = k\ell m^2$ points, and $\mathsf{L}_0 \cup \mathsf{L}_1$ is a lattice with $2M$ points. The tunnels of the i-th layer, joining the i-th and $(i+1)$-st tori, sit over $\mathsf{L}_{(i+1) \bmod 2}$ [22, §1, (2.29)]. Accordingly the set of *removal points* of the j-th torus is

$$Z_1 := \mathsf{L}_0, \qquad Z_N := \mathsf{L}_{N \bmod 2}, \qquad Z_j := \mathsf{L}_0 \cup \mathsf{L}_1 \quad (1 < j < N), \tag{3.6}$$

with $M_j := \#Z_j$ equal to M for $j \in \{1, N\}$ and to $2M$ otherwise, and the *perforated tori* are

$$P_j := \mathbb{T}_{\text{fl}} \setminus \bigcup_{p \in Z_j} B(p, R). \tag{3.7}$$

Since $k \leq \ell$, any two distinct points of $\mathsf{L}_0 \cup \mathsf{L}_1$ are at distance at least $\min\{2Y, \sqrt{X^2 + Y^2}\} \geq Y$, and

$$Y = \frac{\pi}{\sqrt{2}\, \ell m} > \frac{2.2}{\ell m} > 4R, \qquad \frac{R}{2Y} = \frac{1}{10\sqrt{2}\, \pi} < \frac{1}{44}. \tag{3.8}$$

The ratio of the gluing radius R to the lattice spacing is thus a constant independent of m; this is what makes the constants of Section 6 independent of m.

3.4. **Waist radii and balancing conditions.** Let $n := \lfloor N/2 \rfloor$. For $N = 2$ and $N = 3$ set $b_2 := 0$ and $b_2 := 1$ respectively [22, (2.12)]. For $N \geq 4$ the numbers $b_2, \ldots, b_n$ are a solution of the tridiagonal system [22, (3.17)]

$$\begin{aligned} -b_{i-1} + b_2 b_i - b_{i+1} &= -\frac{8\pi}{k\ell m^2}\, b_i \ln b_i, \qquad 2 \leq i \leq n-1, \\ -2^{(N+1) \bmod 2}\, b_{n-1} + \big(b_2 - (N \bmod 2)\big) b_n &= -\frac{8\pi}{k\ell m^2}\, b_n \ln b_n, \end{aligned} \tag{3.9}$$

with $b_1 := 1$; in all cases we extend the family by $b_i := b_{N-i}$ for $n < i \le N-1$, so that $b_1 = b_{N-1} = 1$. We call *limiting system* the system (3.9) with the right-hand sides replaced by 0. From [22, Lemma 3.18 and its proof] we use the following facts.

(B1) For $N \ge 4$ the limiting system has a solution $d = (d_2[N], \dots, d_n[N])$ with $1 < d_2 < d_3 < \cdots < d_n$ and $d_2 \in (1,2)$, and the differential at d of the map $F_N : x \mapsto A_N(x_2)x$ of [22, (3.28)], in terms of which the limiting system reads $F_N(x) = e_1$ [22, (3.21)], is invertible [22, (3.30)].

(B2) For $m \ge m_0[N]$ the numbers $b_i = b_i[N,k,\ell,m]$ are the solution of (3.9) near d furnished by the inverse function theorem, and $b_i \to d_i[N]$ as $m \to \infty$.

The waist radii are defined by [22, (2.13)–(2.14)]: with parameters $\zeta = (\zeta_1, \dots, \zeta_{N-1})$ bounded by a constant $c[N,k,\ell]$ independent of m,

$$\tau_1 = \frac{e^{\zeta_1}}{10\ell m} \exp\Big(-\frac{k\ell m^2}{4\pi}\Big(1 - \frac{b_2}{2}\Big)\Big), \qquad \tau_i = e^{\zeta_i/(k\ell m^2)}\, b_i\, \tau_1 \quad (1 < i < N), \tag{3.10}$$

so that [22, (2.18)]

$$\ln \frac{1}{10\ell m \tau_i} = \frac{k\ell m^2}{4\pi}\Big(1 - \frac{b_2}{2}\Big) - \zeta_1 - (1 - \delta_{i1})\Big(\ln b_i + \frac{\zeta_i}{k\ell m^2}\Big). \tag{3.11}$$

The factors 4π and 8π arise from the linearisation of the minimal surface equation about $\mathbb{T}$, that is from the Jacobi operator (1.2): in the force computation [22, (3.5)–(3.7)] the mean curvature $2\tan 2\mathrm{z} \approx 4\mathrm{z}$ of the parallel torus $\mathbb{T}_{\mathrm{z}}$ is integrated over a cell of area $2\pi^2/(k\ell m^2)$ and balanced against the flux $2\pi\tau$ of a catenoidal neck, and $8\pi^2 = 4 \cdot |\mathbb{T}|$. The heights of the tori and of the necks satisfy [22, (3.6)]

$$\max_j |\mathrm{z}_j| + \max_i |\mathrm{z}_i^K| \le C\, m^2 \tau_1. \tag{3.12}$$

In particular, since $b_2 \le 2 - 1/C$ by [22, (2.12)], τ_1 decays like $\exp(-cm^2)$ with $c = c[N,k,\ell] > 0$, and $m^q \tau_1 \to 0$ for every fixed q; moreover, since $b_i \to d_i[N] > 0$ by (B2) and $|\zeta| \le c$, all the ratios τ_i/τ_1 lie in a compact subset of $(0,\infty)$ depending only on N, for $m \ge m_0$.

3.5. **Catenoidal regions of the initial surface.** Let $\mathbb{K}_a := [-a,a] \times \mathbb{S}^1$ with coordinates (t,θ), $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and flat metric $\widehat{\chi} := dt^2 + d\theta^2$. For $1 \le i \le N-1$ let [22, (4.22)]

$$a_i := \operatorname{arcosh} \frac{1}{10\ell m \tau_i}, \qquad \text{so that} \qquad \tau_i \cosh a_i = R, \tag{3.13}$$

and, by (3.11) and $\operatorname{arcosh} x = \ln x + \ln(1 + \sqrt{1 - x^{-2}})$,

$$a_i = \frac{k\ell m^2}{4\pi}\Big(1 - \frac{b_2}{2}\Big) - \zeta_1 - (1 - \delta_{i1})\Big(\ln b_i + \frac{\zeta_i}{k\ell m^2}\Big) + \ln\big(1 + \sqrt{1 - 100\ell^2 m^2 \tau_i^2}\big). \tag{3.14}$$

The i-th catenoidal region of the initial surface is the image of the map [22, (4.21), (4.23)]

$$\kappa_i : \mathbb{K}_{a_i} \to \mathbb{S}^3, \qquad \kappa_i(t,\theta) := \Phi\big(\mathrm{x}_i + \tau_i \cosh t \cos\theta,\ \mathrm{y}_i + \tau_i \cosh t \sin\theta,\ \mathrm{z}_i^K + \tau_i t\big), \tag{3.15}$$

where $(\mathrm{x}_i, \mathrm{y}_i) = ((i-1)X, (i-1)Y)$ is a point of $\mathsf{L}_{(i+1) \bmod 2}$; the other tunnels of the i-th layer are the images of κ_i under $\mathcal{G}$, and they are pairwise disjoint. The end circles of a tunnel are the images of $\{t = \pm a_i\}$; their projections to the (x,y)-plane are the circles of radius R about the removal point, and the coordinate θ is the polar angle about that point.

Lemma 3.2. *Let g^0 be the metric induced on the initial surface. Then*

$$\kappa_i^* g^0 = \tau_i^2 \cosh^2 t\, \Big(\widehat{\chi} + \sin\big(2\mathrm{z}_i^K + 2\tau_i t\big)\, \mathsf{B}\Big),$$
$$\mathsf{B} := \tanh^2 t \cos 2\theta\, dt^2 - 2\tanh t \sin 2\theta\, dt\, d\theta - \cos 2\theta\, d\theta^2,$$

and consequently, for $m \geq m_0$,

$$(1 - Cm^2\tau_1)\,\tau_i^2 \cosh^2 t\,\widehat{\chi} \;\leq\; \kappa_i^* g^0 \;\leq\; (1 + Cm^2\tau_1)\,\tau_i^2 \cosh^2 t\,\widehat{\chi} \qquad \text{on } \mathbb{K}_{a_i}. \tag{3.16}$$

Proof. Write $\kappa_i = \Phi \circ \widehat{\kappa}_i$ with $\widehat{\kappa}_i(t,\theta) = (\mathrm{x}_i, \mathrm{y}_i, \mathrm{z}_i^K) + \tau_i(\cosh t\cos\theta, \cosh t \sin\theta, t)$. Then $d\mathrm{x} = \tau_i(\sinh t\cos\theta\, dt - \cosh t \sin\theta\, d\theta)$, $d\mathrm{y} = \tau_i(\sinh t \sin\theta\, dt + \cosh t\cos\theta\, d\theta)$, $d\mathrm{z} = \tau_i dt$, so that $d\mathrm{x}^2 + d\mathrm{y}^2 + d\mathrm{z}^2 = \tau_i^2\cosh^2 t\,\widehat{\chi}$ and $d\mathrm{x}^2 - d\mathrm{y}^2 = \tau_i^2\cosh^2 t\,\mathsf{B}$; the formula follows from (3.2) with $\mathrm{z} = \mathrm{z}_i^K + \tau_i t$. The symmetric matrix of B with respect to $\widehat{\chi}$ has entries bounded by 1 in absolute value, so $|\mathsf{B}(v,v)| \leq 2\widehat{\chi}(v,v)$. Finally $|\mathrm{z}_i^K + \tau_i t| \leq |\mathrm{z}_i^K| + \tau_i a_i \leq Cm^2\tau_1$ by (3.12) and (3.14), and $|\sin s| \leq |s|$. ☐

3.6. **Toral regions of the initial surface.** The initial surface $\Sigma^0 = \Sigma^0[N,k,\ell,m,\zeta,\xi]$ is defined in [22, (2.29)–(2.30)] as the $\mathcal{G}$-orbit of N pieces $\Omega_1,\ldots,\Omega_N$, the j-th of which is the image under Φ of the graph, over a cell of the plane $\{\mathrm{z} = \mathrm{z}_j\}$ with one or two discs removed, of the function ϕ of [22, (2.6), (2.9)]. Explicitly, in polar coordinates r about a removal point p of the j-th torus, with τ the waist radius and z^K the height of the neck at p,

$$\phi = \mathrm{z}^K + (\mathrm{z}_j - \mathrm{z}^K)\,\psi[R,2R](r) + \operatorname{sgn}(\mathrm{z}_j - \mathrm{z}^K)\,\tau \operatorname{arcosh}\frac{r}{\tau}\,\psi[2R,R](r) \qquad (\tau \leq r \leq 2R), \tag{3.17}$$

and $\phi = \mathrm{z}_j$ at distance at least $2R$ from all removal points, where $\psi[a,b]$ is a fixed smooth monotone cut-off equal to 0 at a and to 1 at b [22, (2.4)]; for $r \leq R$ the function ϕ is the exact catenoid, and this part is the catenoidal region of the previous subsection. We define the *initial toral region* $\mathcal{T}^0[j]$ to be the part of the j-th torus of Σ^0 lying over P_j, that is the graph of $\phi_j := \phi|_{P_j}$.

Lemma 3.3. *For $m \geq m_0$,*

$$\sup_{P_j} |\phi_j - \mathrm{z}_j| \leq Cm^2\tau_1, \qquad \sup_{P_j}|\nabla\phi_j| \leq Cm^3\tau_1, \qquad \sup_{P_j}|\nabla^2\phi_j| \leq Cm^4\tau_1,$$

and consequently, under the projection $\pi_j : \mathcal{T}^0[j] \to P_j$, $(\mathrm{x},\mathrm{y},\phi_j) \mapsto (\mathrm{x},\mathrm{y})$,

$$(1 - Cm^2\tau_1)\,(d\mathrm{x}^2 + d\mathrm{y}^2) \;\leq\; (\pi_j^{-1})^* g^0 \;\leq\; (1 + Cm^2\tau_1)\,(d\mathrm{x}^2 + d\mathrm{y}^2). \tag{3.18}$$

The map π_j is $\mathcal{G}$-equivariant for the actions of $\mathcal{G}$ on Σ^0 and on $\mathbb{T}_{\mathrm{fl}}$, and it maps each end circle of a tunnel adjoining $\mathcal{T}^0[j]$ onto the circle $\partial B(p,R)$ about the corresponding removal point, the tunnel coordinate θ corresponding to the polar angle.

Proof. On the annulus $R \leq r \leq 2R$ about a removal point, (3.17) gives $|\phi - \mathrm{z}_j| \leq |\mathrm{z}_j - \mathrm{z}^K| + \tau\operatorname{arcosh}(2R/\tau)$. By [22, (2.17)] and (3.12), $|\mathrm{z}_j - \mathrm{z}^K| \leq Cm^2\tau_1$, and $\tau\operatorname{arcosh}(2R/\tau) \leq \tau\ln(4R/\tau) \leq Cm^2\tau_1$ by (3.11). Put $f(r) := \tau\operatorname{arcosh}(r/\tau)$; for $r \geq R \geq 2\tau$ one has $|f| \leq Cm^2\tau_1$, $|f'| = \tau/\sqrt{r^2-\tau^2} \leq 2\tau/R$ and $|f''| = \tau r/(r^2-\tau^2)^{3/2} \leq 2\tau/R^2$, while the cut-offs satisfy $|\psi'| \leq C/R$ and $|\psi''| \leq C/R^2$. Since $R^{-1} = 10\ell m$ and $\tau \leq C\tau_1$, the first derivatives of (3.17) are bounded by $Cm^2\tau_1 \cdot Cm + 2\tau/R \leq Cm^3\tau_1$, and the second derivatives, the Hessian of a radial function $h(r)$ being bounded by $|h''| + |h'|/r$, by $Cm^2\tau_1/R^2 + C\tau/R^2 \leq Cm^4\tau_1$. Elsewhere on P_j, $\phi_j = \mathrm{z}_j$. The induced metric of the graph is, by (3.2),

$$(1 + \sin 2\phi_j)\,d\mathrm{x}^2 + (1 - \sin 2\phi_j)\,d\mathrm{y}^2 + (\partial_{\mathrm{x}}\phi_j\,d\mathrm{x} + \partial_{\mathrm{y}}\phi_j\,d\mathrm{y})^2,$$

which differs from $d\mathrm{x}^2 + d\mathrm{y}^2$ by a form bounded by $(2|\phi_j| + |\nabla\phi_j|^2)(d\mathrm{x}^2 + d\mathrm{y}^2) \leq Cm^2\tau_1(d\mathrm{x}^2 + d\mathrm{y}^2)$. Equivariance holds because Σ^0 is the $\mathcal{G}$-orbit of $\bigcup_j \Omega_j$ and $\mathcal{G}$ acts through Φ by the affine isometries listed after (3.5). The statement about the end circles follows from (3.15) at $t = \pm a_i$, where $\tau_i\cosh a_i = R$. ☐

Lemma 3.4 (Second fundamental form of the initial surface)**.** *Let A^0 be the second fundamental form of Σ^0 and $|A^0|$ its norm with respect to g^0. For $m \geq m_0$,*

$$|A^0| \leq C \quad \text{on } \mathcal{T}^0[j] \quad (1 \leq j \leq N), \qquad |A^0| \circ \kappa_i \leq C\,\tau_i^{-1} \operatorname{sech} t \quad \text{on } \mathbb{K}_{a_i} \quad (1 \leq i \leq N-1).$$

Proof. Let $U := \mathbb{R}^2 \times (-\pi/8, \pi/8)$ and $h := \Phi^* g_{\mathbb{S}^3}|_U = g_E + \sin(2\mathrm{z})(d\mathrm{x}^2 - d\mathrm{y}^2)$, where g_E is the Euclidean metric. Since $|\sin 2\mathrm{z}| \leq \sin(\pi/4) < 3/4$ on U, we have $\frac{1}{4} g_E \leq h \leq 2g_E$, and the Christoffel symbols of h in the coordinates $(\mathrm{x}, \mathrm{y}, \mathrm{z})$, namely $\Gamma^{\mathrm{z}}_{\mathrm{xx}} = -\Gamma^{\mathrm{z}}_{\mathrm{yy}} = -\cos 2\mathrm{z}$, $\Gamma^{\mathrm{x}}_{\mathrm{xz}} = \Gamma^{\mathrm{x}}_{\mathrm{zx}} = \cos 2\mathrm{z}/(1 + \sin 2\mathrm{z})$, $\Gamma^{\mathrm{y}}_{\mathrm{yz}} = \Gamma^{\mathrm{y}}_{\mathrm{zy}} = -\cos 2\mathrm{z}/(1 - \sin 2\mathrm{z})$ and zero otherwise, are bounded by 4. By (3.12), Σ^0 lies in $\Phi(U)$ for $m \geq m_0$, and Φ is a local isometry of (U, h) onto its image.

Let $F : S \to U$ be an immersion of a surface, with unit normals ν_E and ν_h and second fundamental forms A_E and A_h relative to g_E and to h. For tangent vectors V, W, $A_h(V, W) = h\big(\nu_h, \partial_V \partial_W F + \Gamma(\partial_V F, \partial_W F)\big)$, where $\Gamma(\cdot, \cdot)$ denotes the bilinear expression in the Christoffel symbols of h. By the Gauss formula in Euclidean space, $\partial_V \partial_W F = dF(\nabla_V W) + A_E(V, W)\nu_E$ with ∇ the Levi-Civita connection of $F^* g_E$, and ν_h is h-orthogonal to $dF(TS)$; hence

$$A_h(V, W) = A_E(V, W)\, h(\nu_h, \nu_E) + h\big(\nu_h, \Gamma(\partial_V F, \partial_W F)\big).$$

Here $|h(\nu_h, \nu_E)| \leq |\nu_E|_h \leq \sqrt{2}$ and $|h(\nu_h, \Gamma(X, Y))| \leq \sqrt{2}\,|\Gamma(X, Y)|_E \leq C|X|_E |Y|_E$, which is at most $C|X|_h |Y|_h$. Since $F^* g_E$ and $F^* h$ are comparable, $|A_h|_{F^* h} \leq C\big(|A_E|_{F^* g_E} + 1\big)$ with an absolute constant C.

On $\mathcal{T}^0[j]$ take $F(\mathrm{x}, \mathrm{y}) = (\mathrm{x}, \mathrm{y}, \phi_j(\mathrm{x}, \mathrm{y}))$: the induced Euclidean metric is $\geq d\mathrm{x}^2 + d\mathrm{y}^2$ and $A_E = \nabla^2 \phi_j / \sqrt{1 + |\nabla \phi_j|^2}$ in the coordinate frame, so $|A_E| \leq |\nabla^2 \phi_j| \leq C m^4 \tau_1 \leq 1$ by Lemma 3.3. On the i-th catenoidal region take $F = \widehat{\kappa}_i$, the Euclidean catenoid of waist radius τ_i, whose principal curvatures are $\pm \tau_i^{-1} \operatorname{sech}^2 t$, so $|A_E| = \sqrt{2}\,\tau_i^{-1} \operatorname{sech}^2 t \leq \sqrt{2}\,\tau_i^{-1} \operatorname{sech} t$; since $\tau_i^{-1} \operatorname{sech} t \geq \tau_i^{-1} \operatorname{sech} a_i = R^{-1} \geq 1$ on $\mathbb{K}_{a_i}$, the constant 1 is absorbed. ☐

3.7. **The minimal surfaces.** The main theorem of [22] produces, for m large, parameters ζ, ξ bounded by $c[N, k, \ell]$ and a smooth $\mathcal{G}$-invariant function u on $\Sigma^0 = \Sigma^0[N, k, \ell, m, \zeta, \xi]$ whose normal graph

$$\Psi(p) := \exp_p\big(u(p)\, \nu^0(p)\big), \qquad p \in \Sigma^0, \tag{3.19}$$

ν^0 being the unit normal of Σ^0, is a closed embedded minimal surface invariant under $\mathcal{G}$ [22, Theorem 1.1, Lemma 4.40, Theorem 6.49]; the action of $\mathcal{G}$ on normal perturbations carries no sign because every element of $\mathcal{G}$ preserves each side of Σ^0 [22, (4.4)]. We now record the precise form of the estimate on u and the three consequences of it which will be used.

Weighted norms. For a function v on Σ^0, a Riemannian metric h on Σ^0 and a positive weight f, the weighted Hölder norm of [22, (4.6)] is

$$\|v : C^{2,\alpha}(\Sigma^0, h, f)\| := \sup_{p \in \Sigma^0} f(p)^{-1}\, \|v : C^{2,\alpha}(B[p, 1, h], h)\|, \tag{3.20}$$

where $B[p, 1, h]$ is the h-ball of radius 1 about p and the norm on the ball is the sum of the C^2 norm and of the Hölder seminorm of the second derivatives, all measured with h. Let ρ be the $\mathcal{G}$-invariant conformal factor of [22, (4.10)–(4.12)] and let $\chi := \rho^2 g^0$ [22, (4.9)]. The *decay norms* of [22, §4], paragraph "Decay norms and a global estimate of the mean curvature", are defined there, for $\alpha \in (0, 1)$ and $\gamma \in [0, \infty)$, by

$$\|v\|_{2,\alpha,\gamma} := \big\|v : C^{2,\alpha}\big(\Sigma^0, \chi, m^\gamma \rho^{-\gamma}\big)\big\|. \tag{3.21}$$

Since the $C^{2,\alpha}$ norm on the unit χ-ball about p dominates $|v(p)| + |dv|_\chi(p)$, (3.20) and (3.21) give the pointwise bound

$$|v(p)| + |dv|_\chi(p) \;\leq\; m^\gamma \rho(p)^{-\gamma}\, \|v\|_{2,\alpha,\gamma} \qquad (p \in \Sigma^0), \tag{3.22}$$

and, since $\chi = \rho^2 g^0$ and $\chi^{-1} = \rho^{-2}(g^0)^{-1}$ on covectors,

$$|dv|_{g^0} = \rho\, |dv|_\chi \qquad \text{on } \Sigma^0. \tag{3.23}$$

The conformal factor. The function ρ is explicit. By [22, (4.10)–(4.11)] it equals the constant m at distance at least $2R$ from all removal points and, in polar coordinates r about a removal point, $m+(r^{-1}-m)\,\psi[2R,R](r)$ on the transition annulus $R \le r \le 2R$, while on the catenoidal regions [22, (4.25)] $\rho\circ\kappa_i = \tau_i^{-1}\operatorname{sech} t$. Since $r^{-1} \ge (2R)^{-1} = 5\ell m \ge m$ on the annulus,

$$m \le \rho \le R^{-1} = 10\ell m \ \text{ on } \mathcal{T}^0[j], \qquad \rho\circ\kappa_i = \tau_i^{-1}\operatorname{sech} t \ \text{ on } \mathbb{K}_{a_i}. \tag{3.24}$$

On $\mathbb{K}_{a_i}$ one has $\tau_i^{-1}\operatorname{sech} t \ge \tau_i^{-1}\operatorname{sech} a_i = R^{-1} \ge m$ by (3.13), and $\tau_i^{-1}\operatorname{sech} t \le \tau_i^{-1} \le C\tau_1^{-1}$ by the remark after (3.12). Hence, for $m \ge m_0$,

$$m \ \le\ \rho \ \le\ C\,\tau_1^{-1} \qquad \text{on } \Sigma^0, \qquad C = C[N,k,\ell], \tag{3.25}$$

in agreement with [22, (4.13)].

The final estimate. The function u of [22, Theorem 6.49] satisfies [22, (6.60)]

$$\|u\|_{2,\alpha,\gamma} \ \le\ 2C_1\tau_1, \qquad C_1 = C_1[N,k,\ell], \tag{3.26}$$

where $\alpha,\gamma \in (0,1)$ are the exponents fixed in the hypotheses of [22, Theorem 6.49]; the linear theory of [22, §5] is available for every such pair. We stress that u is a function on $\Sigma^0[N,k,\ell,m,\zeta,\xi]$ itself, and that the norm in (3.26) is the decay norm (3.21) of that surface: in the proof of [22, Theorem 6.49] one has $u = u_1 + \mathcal{P}^{-1}v$, where u_1 is the first-order solution of [22, Corollary 5.64] on $\Sigma^0[N,k,\ell,m,\zeta,\xi]$ and v is a fixed point living on the reference surface $\Sigma^0[N,k,\ell,m,0,0]$, transported by the explicit $\mathcal{G}$-equivariant diffeomorphism of [22, (6.41)–(6.43)], whose norm is bounded in [22, Lemma 6.44]; the bound (3.26) is [22, (6.56)]. Of (3.26) we shall use only the pointwise information (3.22) at the level of first derivatives, and of the weight in (3.21) only the positivity of γ; the specific values of α and γ play no role (Lemma 3.7). We retain from all this the following property.

Definition 3.5. Let $N \ge 2$, $k,\ell \ge 1$ and $m \ge m_0[N,k,\ell]$, and let $\Sigma^0 = \Sigma^0[N,k,\ell,m,\zeta,\xi]$ be an initial surface with $|\zeta|,|\xi| \le c[N,k,\ell]$. A closed embedded minimal surface $\Sigma \subset \mathbb{S}^3$ is a *stacked Clifford torus of type* (N,k,ℓ,m) if it is $\mathcal{G}[k,\ell,m]$-invariant and there is a $\mathcal{G}$-equivariant diffeomorphism $\Psi : \Sigma^0 \to \Sigma$ such that the induced metrics g^0 of Σ^0 and g of Σ satisfy

$$\left(1-m^{-2}\right) g^0 \ \le\ \Psi^* g \ \le\ \left(1+m^{-2}\right) g^0 \qquad \text{as quadratic forms on } T\Sigma^0. \tag{W}$$

The verification that the surfaces of [22] have this property rests on two lemmas: a formula for the metric of a normal graph, and a pointwise bound derived from (3.26).

Lemma 3.6 (Metric of a normal graph)**.** *Let $M \subset \mathbb{S}^3$ be a closed embedded surface with induced metric g^0, unit normal ν^0, second fundamental form A^0 and shape operator S, $g^0(SX,Y) = A^0(X,Y)$, and let $u \in C^\infty(M)$. The map $\Psi(p) := \exp_p(u(p)\nu^0(p))$ satisfies*

$$\Psi^* g = \cos^2 u\, g^0 - \sin(2u)\, A^0 + \sin^2 u\, g^0(S\cdot, S\cdot) + du\otimes du, \tag{3.27}$$

g being the round metric. Let $\|S\|$ denote the operator norm of S with respect to g^0 and put $s := |u| + |u|\,\|S\| + |du|_{g^0}$. At every point where $s \le 1$,

$$(1-5s)\, g^0 \ \le\ \Psi^* g \ \le\ (1+5s)\, g^0 \qquad \textit{as quadratic forms.}$$

Proof. Regard $\mathbb{S}^3 \subset \mathbb{R}^4$, write p for the position vector and $\nu^0(p) \in T_p\mathbb{S}^3$ for the unit normal. The geodesics of $\mathbb{S}^3$ being $\tau \mapsto \cos\tau\, p + \sin\tau\, \nu$, we have $\Psi(p) = \cos u(p)\, p + \sin u(p)\, \nu^0(p)$. For $X \in T_pM$, differentiating $\langle \nu^0, p\rangle = 0$ and $\langle \nu^0,\nu^0\rangle = 1$ gives $\langle D_X\nu^0, p\rangle = -\langle \nu^0, X\rangle = 0$ and

$\langle D_X\nu^0,\nu^0\rangle=0$, so $D_X\nu^0\in T_pM$, and $\langle D_X\nu^0,Y\rangle=-\langle\nu^0,D_XY\rangle=-A^0(X,Y)$ for $Y\in T_pM$: thus $D_X\nu^0=-SX$. Hence

$$d\Psi(X)=\cos u\,X-\sin u\,SX+du(X)\,e,\qquad e:=-\sin u\,p+\cos u\,\nu^0,$$

and e is a unit vector orthogonal to T_pM. Expanding $|d\Psi(X)|^2$ gives (3.27). Using $1-\cos^2 u\le u^2$, $|\sin 2u|\le 2|u|$ and $\sin^2 u\le u^2$, (3.27) gives, for every X,

$$\big|\Psi^*g(X,X)-g^0(X,X)\big|\le\big(u^2+2|u|\,\|S\|+u^2\|S\|^2+|du|^2_{g^0}\big)\,g^0(X,X)\le 5s\,g^0(X,X)\qquad\text{if } s\le 1,$$

since then $u^2\le|u|\le s$, $2|u|\,\|S\|\le 2s$, $u^2\|S\|^2\le s^2\le s$ and $|du|^2_{g^0}\le s^2\le s$. ☐

Lemma 3.7 (Pointwise smallness from the weighted estimate)**.** *Let $\Sigma^0=\Sigma^0[N,k,\ell,m,\zeta,\xi]$ with $|\zeta|,|\xi|\le c[N,k,\ell]$ and $m\ge m_0$, and let u be a smooth function on Σ^0 satisfying*

$$\big\|u:C^{2,\alpha}\big(\Sigma^0,\chi,m^\gamma\rho^{-\gamma}\big)\big\|\;\le\;K\,\tau_1$$

for some $\alpha\in(0,1)$, $\gamma>0$ and $K>0$. Then, with s as in Lemma 3.6 for $M=\Sigma^0$,

$$\sup_{\Sigma^0} s\;\le\;C\,K\,m\,\tau_1^{\min\{\gamma,1\}},\qquad C=C[N,k,\ell].$$

Proof. Let $p\in\Sigma^0$ and $f:=m^\gamma\rho^{-\gamma}$. By (3.22) and (3.23), $|u(p)|\le f(p)K\tau_1$ and $|du|_{g^0}(p)\le\rho(p)f(p)K\tau_1$. By Lemma 3.4 and (3.24), $\|S\|\le|A^0|\le C_A\,\rho$ on all of Σ^0 with $C_A=C_A[N,k,\ell]$: on the toral regions $|A^0|\le C\le C\rho$ because $\rho\ge m\ge 1$, and on the catenoidal regions $|A^0|\le C\tau_i^{-1}\operatorname{sech}t=C\rho$. Hence, using $\rho\ge 1$,

$$s(p)\le\big(1+C_A\rho(p)+\rho(p)\big)f(p)K\tau_1\le(2+C_A)\,\rho(p)f(p)\,K\tau_1,\qquad \rho f=m^\gamma\rho^{1-\gamma}.$$

If $\gamma\ge 1$, then $\rho^{1-\gamma}\le m^{1-\gamma}$ by the lower bound in (3.25), so $\rho f\le m$ and $s(p)\le(2+C_A)Km\tau_1$. If $0<\gamma<1$, then $\rho^{1-\gamma}\le C^{1-\gamma}\tau_1^{\gamma-1}$ by the upper bound in (3.25), so $\rho f\,\tau_1\le C\,m^\gamma\tau_1^\gamma\le C\,m\,\tau_1^\gamma$ and $s(p)\le(2+C_A)CKm\tau_1^\gamma$. In both cases $s(p)\le CKm\tau_1^{\min\{\gamma,1\}}$. ☐

Corollary 3.8 (The surfaces of [22] are stacked Clifford tori)**.** *Let Σ be the normal graph* (3.19) *over $\Sigma^0=\Sigma^0[N,k,\ell,m,\zeta,\xi]$, $|\zeta|,|\xi|\le c[N,k,\ell]$, of the smooth $\mathcal{G}$-invariant function u produced by* [22, Theorem 6.49]*, which satisfies* (3.26)*. Then, for $m\ge m_0[N,k,\ell]$, Σ is a stacked Clifford torus of type (N,k,ℓ,m), with Ψ as in* (3.19)*.*

Proof. The estimate. By Lemma 3.7 with $K=2C_1$, $\sup_{\Sigma^0}s\le Cm\tau_1^{\min\{\gamma,1\}}$. By (3.10) and the remark after (3.12), $\tau_1\le\exp(-cm^2)$ with $c=c[N,k,\ell]>0$ for $m\ge m_0$, so $5\sup s\le 5Cm\exp(-c\min\{\gamma,1\}m^2)\le m^{-2}$ after enlarging m_0. In particular $s\le 1$ everywhere, and Lemma 3.6 gives (W).

Equivariance and bijectivity. For $\mathfrak{g}\in\mathcal{G}$ the function u is invariant and, by [22, (4.4)], $\mathfrak{g}$ preserves each side of Σ^0, that is $\mathfrak{g}_*\nu^0(p)=\nu^0(\mathfrak{g}p)$; as $\mathfrak{g}$ is linear on $\mathbb{R}^4$, $\Psi(\mathfrak{g}p)=\cos u(p)\,\mathfrak{g}p+\sin u(p)\,\mathfrak{g}\nu^0(p)=\mathfrak{g}\Psi(p)$. By (W), Ψ is an immersion of the closed surface Σ^0. The map Ψ is the normal deformation $\iota[u]$ of [22, (5.1)], and the last paragraph of the proof of [22, Theorem 6.49] shows that $\iota[u]$ is an embedding for $m\ge m_0$: the decay of u toward the waists built into (3.21) keeps $|u|$ below a small multiple of τ_1 on the catenoidal cores, hence below the normal injectivity radius of Σ^0 there. Thus Ψ is injective, and it is a diffeomorphism onto its image, the embedded surface Σ. Finally Σ is closed, embedded, minimal and $\mathcal{G}$-invariant by the same theorem. ☐

Remark 3.9. Whether the fixed point of [22, Theorem 6.49] is unique plays no role: Theorem A applies to every surface with the property of Definition 3.5, and Corollary 3.8 applies to every fixed point. The bound (3.26) is far stronger than needed: the metric distortion it produces is of order $m\tau_1^{\min\{\gamma,1\}}$, whereas (W) only asks for m^{-2}. Moreover Lemma 3.7 uses of the weight in (3.21) only the positivity of the exponent γ, through the two-sided bound (3.25): for $\gamma = 0$ the argument would give no more than $s \le C\rho\tau_1$, which is of order one on the part $|t| \lesssim \ln m$ of each tunnel. The decay of u toward the waists built into (3.21) is what makes the sharp comparison (W) available on all of Σ, and we use it in that form throughout. (We expect the leading value 4 of Theorem B to survive a bounded distortion of the metric on a portion of each tunnel of conformal length $O(\ln m)$, since such a portion would contribute only $O(m^{-2}\ln m)$ to the reciprocal of the conductance of a tunnel of conformal length $2a_i \sim m^2$; we do not pursue this, as (W) is available.)

3.8. **Decomposition and comparison.** Let Σ be a stacked Clifford torus of type (N, k, ℓ, m) with Ψ as in (W). We transport the decomposition of Σ^0 to Σ:

$$\Sigma = \bigcup_{j=1}^{N} \mathcal{T}[j] \;\cup\; \bigcup_{i=1}^{N-1} \bigcup_{s=1}^{M} \mathcal{K}_i^{(s)}, \qquad \mathcal{T}[j] := \Psi(\mathcal{T}^0[j]), \quad \mathcal{K}_i^{(s)} := \Psi\big(\mathfrak{g}_s \kappa_i(\mathbb{K}_{a_i})\big), \tag{3.28}$$

where $\mathfrak{g}_1 = \mathrm{id}, \mathfrak{g}_2, \dots, \mathfrak{g}_M \in \mathcal{G}$ carry the removal point $(\mathrm{x}_i, \mathrm{y}_i)$ onto the M points of $\mathsf{L}_{(i+1) \bmod 2}$. The pieces have pairwise disjoint interiors and are glued along the end circles $\Gamma_{i,\pm}^{(s)} := \Psi(\mathfrak{g}_s\kappa_i(\{t = \pm a_i\}))$. Each tunnel $\mathcal{K}_i^{(s)}$ carries the coordinates $(t,\theta) \in \mathbb{K}_{a_i}$ through $\Psi \circ \mathfrak{g}_s \circ \kappa_i$, and we write $\mathcal{K}_i^{(s),\pm}$ for its halves $\{\pm t \ge 0\}$. Each toral region $\mathcal{T}[j]$ carries the chart $\pi_j \circ \Psi^{-1} : \mathcal{T}[j] \to P_j$. By Lemmas 3.2, 3.3 and (W), for $m \ge m_0$ (so that $Cm^2\tau_1 \le m^{-2}$),

$$\begin{aligned} &(1-\varepsilon_m)\,\tau_i^2\cosh^2 t\,\widehat{\chi} \le g \le (1+\varepsilon_m)\,\tau_i^2\cosh^2 t\,\widehat{\chi} \quad \text{on } \mathcal{K}_i^{(s)},\\ &(1-\varepsilon_m)\,(\mathrm{d}\mathrm{x}^2+\mathrm{d}\mathrm{y}^2) \le g \le (1+\varepsilon_m)\,(\mathrm{d}\mathrm{x}^2+\mathrm{d}\mathrm{y}^2) \quad \text{on } \mathcal{T}[j], \end{aligned} \qquad \varepsilon_m := 3m^{-2}. \tag{3.29}$$

All these identifications are $\mathcal{G}$-equivariant, $\mathcal{G}$ preserving each $\mathcal{T}[j]$ and permuting the tunnels of each layer among themselves.

Convention 3.10. For an end circle $\Gamma = \Gamma_{i,\pm}^{(s)}$ and a function u on Σ we write

$$\operatorname{avg}_\Gamma u := \frac{1}{2\pi}\int_0^{2\pi} u(\pm a_i,\theta)\,d\theta$$

for the mean with respect to the coordinate θ; by Lemma 3.3 this is also the mean over the circle $\partial B(p,R)$ of P_j with respect to the polar angle. For a region $\Omega \subset \Sigma$ we write $\operatorname{avg}_\Omega u := |\Omega|^{-1}\int_\Omega u\,dA$ with respect to the area of Σ.

Lemma 3.11 (Transfer)**.** *Let h and h' be Riemannian metrics on a surface S with $(1-\varepsilon)h \le h' \le (1+\varepsilon)h$, $0 < \varepsilon \le \frac{1}{2}$. Then for every $f \in H^1(S)$*

$$\frac{1-\varepsilon}{1+\varepsilon}\int_S |\nabla f|_h^2\,dA_h \le \int_S |\nabla f|_{h'}^2\,dA_{h'} \le \frac{1+\varepsilon}{1-\varepsilon}\int_S |\nabla f|_h^2\,dA_h,$$
$$(1-\varepsilon)\,dA_h \le\, dA_{h'} \le (1+\varepsilon)\,dA_h,$$

and in dimension two the Dirichlet integral $\int |\nabla f|_h^2 dA_h$ is invariant under conformal changes of h. In particular all comparisons in (3.29) affect Dirichlet integrals, L^2 norms and areas by factors $1 + O(m^{-2})$.

Proof. If $h' \le (1+\varepsilon)h$ then $h'^{-1} \ge (1+\varepsilon)^{-1}h^{-1}$ on covectors, so $|\nabla f|_{h'}^2 \ge (1+\varepsilon)^{-1}|\nabla f|_h^2$, while $dA_{h'} \ge (1-\varepsilon)dA_h$ from $h' \ge (1-\varepsilon)h$; the other inequalities are symmetric. Conformal invariance in dimension two is the identity $|\nabla f|_{\rho^2 h}^2 dA_{\rho^2 h} = |\nabla f|_h^2 dA_h$. $\square$

4. Balancing, the line graph and the spectral gap

Let P_N be the path graph with vertices $1,\dots,N$ and edges $e_i=\{i,i+1\}$, $1\le i\le N-1$; let $A(P_N)$ be its adjacency matrix and $L(P_N)=\mathrm{diag}(\deg)-A(P_N)$ its combinatorial Laplacian, whose quadratic form is $c\mapsto\sum_{i=1}^{N-1}(c_i-c_{i+1})^2$. The line graph of P_N, whose vertices are the edges $e_1,\dots,e_{N-1}$ with e_i adjacent to e_{i+1}, is P_{N-1}. Let D be the $N\times(N-1)$ incidence matrix with the consistent orientation, $D_{v,e_i}=-1$ if $v=i$, $D_{v,e_i}=+1$ if $v=i+1$, and $D_{v,e_i}=0$ otherwise.

Lemma 4.1 (Incidence identity)**.** *$L(P_N)=DD^\mathsf{T}$ and $D^\mathsf{T}D=2I_{N-1}-A(P_{N-1})$. The matrix D has rank $N-1$, the matrix $D^\mathsf{T}D$ is positive definite, and its spectrum, with multiplicities, is the spectrum of $L(P_N)$ with the eigenvalue 0 removed. Consequently*

$$\lambda_1\big(L(P_N)\big)=2-\lambda_{\max}\big(A(P_{N-1})\big),\qquad \lambda_{\max}\big(L(P_N)\big)=2-\lambda_{\min}\big(A(P_{N-1})\big).$$

Proof. $(DD^\mathsf{T})_{vw}=\sum_e D_{ve}D_{we}$ equals the number of edges at v if $v=w$, equals $(-1)(+1)=-1$ if $\{v,w\}$ is an edge, and vanishes otherwise; this is $L(P_N)$. $(D^\mathsf{T}D)_{ef}=\sum_v D_{ve}D_{vf}$ equals 2 if $e=f$; if $e=e_i$ and $f=e_{i+1}$ the only common vertex is $i+1$, which is the head of e_i and the tail of e_{i+1}, so the entry is $(+1)(-1)=-1$; and it vanishes if e and f are not adjacent. Thus $D^\mathsf{T}D=2I-A(P_{N-1})$. Next, $D^\mathsf{T}c=0$ means $c_{i+1}=c_i$ for all i, so $\ker D^\mathsf{T}$ consists of the constant vectors and $\operatorname{rank}D=N-1$; hence $\ker(DD^\mathsf{T})=\ker D^\mathsf{T}$ is one-dimensional and $D^\mathsf{T}D$ is positive definite. For any real matrix the nonzero eigenvalues of DD^T and $D^\mathsf{T}D$ coincide with multiplicities (they are the squares of the nonzero singular values of D); as DD^T has exactly $N-1$ nonzero eigenvalues counted with multiplicity, these are all the eigenvalues of the $(N-1)\times(N-1)$ matrix $D^\mathsf{T}D$. The last two identities follow. □

Lemma 4.2 (Spectra of paths)**.** (i) *For $M\ge1$ the eigenvalues of $A(P_M)$ are $2\cos\big(j\pi/(M+1)\big)$, $j=1,\dots,M$, with eigenvectors $v^{(j)}=\big(\sin(ij\pi/(M+1))\big)_{i=1}^M$; for $j=1$ the eigenvector is positive.*

(ii) *The eigenvalues of $L(P_N)$ are $2\big(1-\cos(j\pi/N)\big)$, $j=0,\dots,N-1$, with eigenvectors $c^{(j)}=\big(\cos\frac{(2r-1)j\pi}{2N}\big)_{r=1}^N$. In particular $\lambda_1(L(P_N))=2(1-\cos(\pi/N))$ is attained at $c^{(1)}$, which satisfies $c^{(1)}_{N+1-r}=-c^{(1)}_r$ and $\sum_r c^{(1)}_r=0$.*

Proof. (i) With $x_i:=ij\pi/(M+1)$ and $y:=j\pi/(M+1)$, the identity $\sin(x_i-y)+\sin(x_i+y)=2\sin x_i\cos y$ gives $(Av^{(j)})_i=v^{(j)}_{i-1}+v^{(j)}_{i+1}=2\cos y\,v^{(j)}_i$ for $1<i<M$, and also for $i=1$ and $i=M$ because $\sin(0)=0=\sin(j\pi)$. The M numbers $2\cos(j\pi/(M+1))$ are distinct, so these are all the eigenvalues, and $v^{(1)}_i=\sin(i\pi/(M+1))>0$.

(ii) With $x_r:=(2r-1)j\pi/(2N)$ and $y:=j\pi/N$, the identity $\cos(x_r-y)+\cos(x_r+y)=2\cos x_r\cos y$ gives $-c^{(j)}_{r-1}+2c^{(j)}_r-c^{(j)}_{r+1}=2(1-\cos y)c^{(j)}_r$ for $1<r<N$. At $r=1$ the row of $L(P_N)$ is $(1,-1,0,\dots)$ and the same identity applies because $c^{(j)}_0=\cos(-j\pi/2N)=c^{(j)}_1$; at $r=N$ it applies because $c^{(j)}_{N+1}=\cos(j\pi+j\pi/2N)=(-1)^j\cos(j\pi/2N)=c^{(j)}_N$. The N eigenvalues $2(1-\cos(j\pi/N))$, $0\le j\le N-1$, are distinct, so the $c^{(j)}$ form an eigenbasis. Finally $c^{(1)}_{N+1-r}=\cos\big(\pi-\frac{(2r-1)\pi}{2N}\big)=-c^{(1)}_r$, whence $\sum_r c^{(1)}_r=0$. □

Lemma 4.3 (Perron structure of the limiting system)**.** *Let $N\ge2$ and let $b=(b_1,\dots,b_{N-1})$ be given by $b_1=1$, by a solution $(b_2,\dots,b_n)$ of the limiting system with all $b_i>0$ when $N\ge4$, by the conventions $b_2=0$ $(N=2)$, $b_2=1$ $(N=3)$, and by $b_i=b_{N-i}$ for $n<i\le N-1$. Then b is a positive vector and*

$$A(P_{N-1})\,b=b_2\,b.$$

Proof. Set $b_0 := b_N := 0$, so that $(A(P_{N-1})b)_i = b_{i-1} + b_{i+1}$ for $1 \le i \le N-1$. For $N = 2$ the graph P_1 has a single vertex, $A(P_1) = 0$, $b = (1)$ and $b_2 = 0$. For $N = 3$, $b = (1, 1)$ and $A(P_2)b = (1, 1) = b_2 b$. Let $N \ge 4$, so $n \ge 2$. For $i = 1$: $b_0 + b_2 = b_2 = b_2 b_1$. For $2 \le i \le n-1$: $b_{i-1} + b_{i+1} = b_2 b_i$ is the first line of the limiting system. For $i = n$: if $N = 2n$ then $b_{n+1} = b_{N-n-1} = b_{n-1}$ and the second line reads $2b_{n-1} = b_2 b_n$; if $N = 2n+1$ then $b_{n+1} = b_{N-n-1} = b_n$ and the second line reads $b_{n-1} + b_n = b_2 b_n$. For $n < i \le N-1$ we have $1 \le N-i \le n$ and, by the symmetry $b_i = b_{N-i}$, $b_{i-1} + b_{i+1} = b_{N-i+1} + b_{N-i-1} = b_2 b_{N-i} = b_2 b_i$ by the cases already treated. Positivity holds because $b_1 = 1$ and $b_i > 0$ for $2 \le i \le n$. ☐

Proposition 4.4 (Closed form of the balancing constants)**.** *Let $N \ge 2$ and let $d_i[N]$ be the limiting waist ratios of* (B1)*, with $d_2[2] = 0$ and $d_2[3] = 1$, extended by $d_1 = 1$ and $d_i = d_{N-i}$. Then*

$$d_i[N] = \frac{\sin(i\pi/N)}{\sin(\pi/N)} \quad (1 \le i \le N-1),$$

$$b_2[N] := d_2[N] = 2\cos\frac{\pi}{N} = \lambda_{\max}\big(A(P_{N-1})\big), \qquad \lambda_1\big(L(P_N)\big) = 2 - b_2[N].$$

In particular $b_2[4] = \sqrt{2}$ and $b_2[5] = \frac{1+\sqrt{5}}{2}$.

Proof. For $N = 2$ and $N = 3$ the statements are immediate from $A(P_1) = 0$ and $\operatorname{spec} A(P_2) = \{\pm 1\}$. Let $N \ge 4$. By Lemma 4.3 the vector $d = (d_1, \dots, d_{N-1})$ is a positive eigenvector of $A(P_{N-1})$ with eigenvalue d_2. The matrix $A(P_{N-1})$ is nonnegative and irreducible, P_{N-1} being connected, so by the Perron–Frobenius theorem it has, up to positive multiples, a unique positive eigenvector, whose eigenvalue is the spectral radius $\lambda_{\max}(A(P_{N-1}))$. Hence $d_2 = \lambda_{\max}(A(P_{N-1})) = 2\cos(\pi/N)$ by Lemma 4.2(i) with $M = N-1$, and d is the positive multiple of $\big(\sin(i\pi/N)\big)_i$ with first entry 1. The identity for $\lambda_1(L(P_N))$ is Lemma 4.1. The special values are $2\cos(\pi/4) = \sqrt{2}$ and $2\cos(\pi/5) = (1+\sqrt{5})/2$. ☐

Corollary 4.5. *For $m \ge m_0[N]$ the constants of* (3.9) *satisfy*

$$|b_i[N,k,\ell,m] - d_i[N]| \le \frac{C}{m^2} \quad (1 \le i \le N-1), \qquad 2 - b_2[N,k,\ell,m] \;\ge\; 1 - \cos\frac{\pi}{N} \;> 0,$$

and all b_i lie in a compact subset of $(0,\infty)$ depending only on N.

Proof. For $N \in \{2, 3\}$ the constants are exact. Let $N \ge 4$, let $A_N(\beta)$ be the $(n-1)\times(n-1)$ matrix of [22, (3.19)–(3.20)], and put $F(x) := A_N(x_2)x - e_1$ and $G(x) := (x_i \ln x_i)_{i=2}^n$ for $x = (x_2, \dots, x_n)$ with positive entries. The limiting system is $F(x) = 0$, and (3.9) is $F(x) + \varepsilon G(x) = 0$ with $\varepsilon := 8\pi/(k\ell m^2)$. By (B1), $F(d) = 0$ and $dF(d)$ is invertible; the map $H(x,\varepsilon) := F(x) + \varepsilon G(x)$ is smooth near $(d, 0)$ with $\partial_x H(d,0) = dF(d)$, so the implicit function theorem gives a smooth branch $x(\varepsilon)$ of solutions with $x(0) = d$, unique in a neighbourhood of d, and $|x(\varepsilon) - d| \le C[N]\varepsilon$ for small ε. By (B2), $b[N,k,\ell,m] = x(\varepsilon)$ for $m \ge m_0[N]$, which gives the first estimate for $2 \le i \le n$, hence for all i by the symmetry. Since $2 - d_2[N] = 2(1 - \cos(\pi/N))$ by Proposition 4.4, the second estimate holds for m large, and the last statement follows from $d_i[N] > 0$. ☐

Remark 4.6 (The heights form the Fiedler vector)**.** With the consistent orientation, $(Db)_v = b_{v-1} - b_v$, and for the limiting ratios $b_i = \sin(i\pi/N)/\sin(\pi/N)$ one computes

$$(Db)_v = \frac{\sin\frac{(v-1)\pi}{N} - \sin\frac{v\pi}{N}}{\sin\frac{\pi}{N}} = -\frac{\cos\frac{(2v-1)\pi}{2N}}{\cos\frac{\pi}{2N}} = -\frac{c_v^{(1)}}{\cos\frac{\pi}{2N}}.$$

Thus Db is the eigenvector of $L(P_N) = DD^{\mathsf{T}}$ for its smallest nonzero eigenvalue, as it must be: $DD^{\mathsf{T}}(Db) = D(D^{\mathsf{T}}Db) = (2 - b_2)Db$. In the linear model of the construction the vector of heights of the tori is proportional to Db [22, (2.15)–(2.17)], so the stacking configuration is, to leading order, the Fiedler vector of the chain. This is the geometric reason why the mode which is constant on each torus is governed by the spectral gap and not by another eigenvalue of $L(P_N)$.

Remark 4.7 (Chebyshev polynomials)**.** In [22, Lemma 3.18] the number $d_2[N]$ is characterised as the largest root of a polynomial P_{n+1} defined by the recursion $P_{i+1}(\lambda) = \lambda P_i(\lambda) - P_{i-1}(\lambda)$ with initial data $P_1 = 2$, $P_2 = \lambda$ for even N and $P_1 = 1$, $P_2 = \lambda - 1$ for odd N [22, (3.22)–(3.25)]. This is the Chebyshev recursion, and the substitution $\lambda = 2\cos\theta$ gives $P_i(2\cos\theta) = 2\cos((i-1)\theta)$ in the even case and $P_i(2\cos\theta) = \cos\big(\frac{(2i-1)\theta}{2}\big)/\cos\frac{\theta}{2}$ in the odd case, as one checks from $2\cos\theta\cos\alpha = \cos(\alpha+\theta) + \cos(\alpha-\theta)$ and the initial data. The largest root of P_{n+1} is therefore $2\cos\theta_0$ with $\theta_0 = \pi/(2n) = \pi/N$ for $N = 2n$ and $\theta_0 = \pi/(2n+1) = \pi/N$ for $N = 2n+1$, in agreement with Proposition 4.4, which also explains the monotonicity of $d_2[N]$ in N observed in [22]. The closed form of all the ratios $d_i[N]$, and their interpretation as the Perron vector of the line graph, do not seem to have been recorded before.

5. The channel inequality and the conductances

Lemma 5.1 (Channel inequality)**.** *Let $a > 0$, let $\mathbb{K}_a = [-a, a] \times \mathbb{S}^1$ with the flat metric $\widehat{\chi} = dt^2 + d\theta^2$, and let $\Gamma_\pm = \{t = \pm a\}$. For every $f \in H^1(\mathbb{K}_a)$,*

$$\int_{\mathbb{K}_a} |\nabla f|^2\, dt\, d\theta \;\geq\; \frac{\pi}{a}\Big(\operatorname{avg}_{\Gamma_+} f - \operatorname{avg}_{\Gamma_-} f\Big)^2, \qquad \operatorname{avg}_{\Gamma_\pm} f := \frac{1}{2\pi}\int_0^{2\pi} f(\pm a, \theta)\, d\theta,$$

with equality if and only if $f(t,\theta) = \alpha + \beta t$ for constants α, β.

Proof. Both sides are continuous on $H^1(\mathbb{K}_a)$, the traces on $\Gamma_\pm$ being continuous, so it suffices to consider smooth f. For each θ, $f(a,\theta) - f(-a,\theta) = \int_{-a}^{a} f_t\, dt$, so by the Cauchy–Schwarz inequality

$$\big(f(a,\theta) - f(-a,\theta)\big)^2 \leq 2a \int_{-a}^{a} f_t(t,\theta)^2\, dt. \tag{5.1}$$

With $\delta := \operatorname{avg}_{\Gamma_+} f - \operatorname{avg}_{\Gamma_-} f = \frac{1}{2\pi}\int_0^{2\pi}(f(a,\theta) - f(-a,\theta))\, d\theta$, a second application gives

$$\delta^2 \leq \frac{1}{2\pi}\int_0^{2\pi} \big(f(a,\theta) - f(-a,\theta)\big)^2\, d\theta. \tag{5.2}$$

Combining, $\delta^2 \leq \frac{a}{\pi}\int_{\mathbb{K}_a} f_t^2 \leq \frac{a}{\pi}\int_{\mathbb{K}_a} |\nabla f|^2$. Equality forces f_t to be constant in t for almost every θ (equality in (5.1)), $f(a,\theta) - f(-a,\theta)$ to be constant in θ (equality in (5.2)), and $f_\theta \equiv 0$ (equality in the last step); together these give $f = \alpha + \beta t$. Conversely, for $f = t/a$ one has $\delta = 2$ and $\int |\nabla f|^2 = a^{-2} \cdot 2a \cdot 2\pi = (\pi/a)\delta^2$. ☐

Corollary 5.2. *Let g be a metric on $\mathbb{K}_a$ with $(1-\varepsilon)\rho^2\widehat{\chi} \leq g \leq (1+\varepsilon)\rho^2\widehat{\chi}$ for a positive function ρ and some $\varepsilon \in (0, \frac{1}{2}]$. Then for every $f \in H^1(\mathbb{K}_a)$*

$$\int_{\mathbb{K}_a} |\nabla f|_g^2\, dA_g \;\geq\; \frac{1-\varepsilon}{1+\varepsilon} \cdot \frac{\pi}{a}\Big(\operatorname{avg}_{\Gamma_+} f - \operatorname{avg}_{\Gamma_-} f\Big)^2.$$

Proof. By Lemma 3.11 the Dirichlet integral of g is at least $\frac{1-\varepsilon}{1+\varepsilon}$ times that of $\rho^2\widehat{\chi}$, which equals that of $\widehat{\chi}$ by conformal invariance. ☐

The channel inequality plays the role that the capacity of a tunnel would play, but it is an inequality for arbitrary H^1 functions with no condition on the boundary data, and it is sharp; its conformal invariance is what allows the geometry of a tunnel to enter only through the single number a.

Definition 5.3. We call π/a the *conductance* of a channel of conformal length $2a$, and for a stacked Clifford torus of type (N,k,ℓ,m) we set

$$C_i := M\,\frac{\pi}{a_i} = \frac{k\ell m^2\pi}{a_i}, \qquad 1\le i\le N-1,$$

the total conductance of the M tunnels joining the i-th and the $(i+1)$-st tori.

Proposition 5.4. *For $m\ge m_0$ and every $1\le i\le N-1$,*

$$C_i = \frac{8\pi^2}{2-b_2[N,k,\ell,m]}\big(1+O(m^{-2})\big) = \frac{8\pi^2}{2-b_2[N]}\big(1+O(m^{-2})\big) = \frac{4\pi^2}{1-\cos(\pi/N)}\big(1+O(m^{-2})\big).$$

In particular C_i is bounded above and below by positive constants depending only on N, the total conductance is to leading order the same for all layers and independent of k, ℓ and m, and $C_N := 8\pi^2/(2-b_2[N])$ satisfies $C_N = 8N^2+O(1)$ as $N\to\infty$.

Proof. In (3.14) the terms ζ_1, $\ln b_i$, $\zeta_i/(k\ell m^2)$ and $\ln(1+\sqrt{1-100\ell^2m^2\tau_i^2})\in[0,\ln 2]$ are bounded by constants depending only on N,k,ℓ, so $a_i = \frac{k\ell m^2}{8\pi}(2-b_2)+O(1)$, and since $2-b_2\ge 1-\cos(\pi/N)$ by Corollary 4.5, $a_i = \frac{k\ell m^2}{8\pi}(2-b_2)(1+O(m^{-2}))$. Hence $C_i = k\ell m^2\pi/a_i = 8\pi^2(2-b_2)^{-1}(1+O(m^{-2}))$. Corollary 4.5 gives $2-b_2 = (2-b_2[N])(1+O(m^{-2}))$, and $2-b_2[N] = 2(1-\cos(\pi/N))$ by Proposition 4.4. Finally $1-\cos(\pi/N) = \frac{\pi^2}{2N^2}-\frac{\pi^4}{24N^4}+O(N^{-6})$ gives $C_N = 8N^2\big(1-\frac{\pi^2}{12N^2}+O(N^{-4})\big)^{-1} = 8N^2+O(1)$. □

Remark 5.5 (The regime $m\gg N$)**.** By Proposition 4.4, $2-b_2[N] = 2(1-\cos(\pi/N)) = \pi^2N^{-2}(1+O(N^{-2}))$, so that (3.14) reads

$$a_i = \frac{k\ell\pi m^2}{8N^2}\big(1+O(N^{-2})\big)+O(1),$$

the $O(1)$ term collecting $-\zeta_1-(1-\delta_{i1})(\ln b_i+\zeta_i/(k\ell m^2))+\ln(1+\sqrt{1-100\ell^2m^2\tau_i^2})$ and the correction $b_2-b_2[N] = O(m^{-2})$. The construction of [22] requires $a_i\to\infty$, that is tunnels which are long catenoids, so the hypothesis $m\ge m_0[N,k,\ell]$ entails $m\gg N/\sqrt{k\ell}$ already at the level of the initial surfaces. Accordingly the relative error $O(m^{-2})$ of Proposition 5.4 is of size $O(N^2/(k\ell m^2))$, with the corresponding growth in N of the constants of Lemma 6.8 and Proposition 6.12 below and of the threshold m_0 in Theorem A. The leading value 4 of Theorem B is the only quantity which is uniform in N.

6. The invariant sector

Throughout this section Σ is a stacked Clifford torus of type (N,k,ℓ,m) with $m\ge m_0$, decomposed as in (3.28), and $\mathcal{G} = \mathcal{G}[k,\ell,m]$. Since every element of $\mathcal{G}$ preserves each side of the tori, $\mathcal{G}$ acts on functions on Σ by composition, and we set

$$H^1(\Sigma)^{\mathcal{G}} := \big\{u\in H^1(\Sigma) : u\circ\mathfrak{g} = u \text{ for all } \mathfrak{g}\in\mathcal{G}\big\}.$$

The group $\mathcal{G}$ maps each toral region onto itself and permutes the tunnels of each layer among themselves. Moreover every $\mathfrak{g}\in\mathcal{G}$ carries the tunnel of the i-th layer over a removal point p onto the tunnel of the same layer over $\mathfrak{g}p$ by a map which, in the tunnel coordinates (3.15), has the form $(t,\theta)\mapsto(t,\pm\theta+\theta_0)$: indeed, by the description of the action following (3.5), $\mathfrak{g}$ fixes the coordinate z and acts on (x,y) by an affine isometry whose linear part is diagonal with entries ±1, so that it preserves the height $\mathrm{z}_i^K+\tau_i t$, hence t, and rotates or reflects the polar angle θ about the removal point. In particular a function which is constant on each toral region and depends only on t and on the layer in each tunnel is $\mathcal{G}$-invariant. For $u \in H^1(\Sigma)$ we write

$$c_j := \operatorname{avg}_{\mathcal{T}[j]} u, \qquad d_i := c_{i+1} - c_i, \qquad \mathcal{E}_{\mathrm{tor}} := \sum_{j=1}^{N} \int_{\mathcal{T}[j]} |\nabla u|^2, \qquad \mathcal{E}_{\mathrm{tun}} := \sum_{i=1}^{N-1} \sum_{s=1}^{M} \int_{\mathcal{K}_i^{(s)}} |\nabla u|^2,$$

all integrals being taken with respect to the area of Σ unless a flat measure $dx\,dy$ or $dt\,d\theta$ is indicated. Finally we set

$$A_j := |\mathcal{T}[j]|, \qquad \Theta_j := M_j\,\pi R^2, \qquad \widetilde{A}_j := A_j + \Theta_j, \qquad \vartheta_j := \frac{\Theta_j}{2\pi^2} = \frac{M_j}{2M} \cdot \frac{k}{100\pi\ell} \le \frac{1}{100\pi}, \tag{6.1}$$

where $M_j = \#Z_j \in \{M, 2M\}$ is the number of tunnels adjoining $\mathcal{T}[j]$; by (3.7), $|P_j| = 2\pi^2(1-\vartheta_j)$ and $\Theta_j = |\mathbb{T}_{\mathrm{fl}} \setminus P_j|$.

6.1. Areas.

Lemma 6.1. *For $m \ge m_0$, every tunnel half and every j,*

$$\big|\mathcal{K}_i^{(s),\pm}\big| = \pi R^2\big(1 + O(m^{-2})\big) \le 2\pi R^2, \qquad A_j = |P_j|\big(1 + O(m^{-2})\big),$$
$$\widetilde{A}_j = 2\pi^2\big(1 + O(m^{-2})\big), \qquad \pi^2 \le A_j \le \widetilde{A}_j \le 3\pi^2.$$

In particular, for each layer i, $M|\mathcal{K}_i^{(s),\pm}| = \pi R^2 M(1 + O(m^{-2}))$: the halves of the tunnels of one layer have, to leading order, exactly the area $\pi R^2 M$ removed from each of the two tori they join.

Proof. In the model metric $\tau_i^2 \cosh^2 t\,\widehat{\chi}$ the area of $\{0 \le t \le a_i\}$ is $2\pi\tau_i^2 \int_0^{a_i} \cosh^2 t\,dt = \pi\tau_i^2(a_i + \sinh a_i \cosh a_i) = \pi R^2(\tanh a_i + a_i \operatorname{sech}^2 a_i)$, using $\tau_i \cosh a_i = R$. Since $a_i \ge cm^2$ by (3.14) and Corollary 4.5, and $1 - \tanh a \le 2e^{-2a}$, $a \operatorname{sech}^2 a \le 4ae^{-2a}$, this is $\pi R^2(1 + O(m^{-2}))$; by (3.29) and Lemma 3.11 the area in Σ differs from it by a factor $1 + O(m^{-2})$. Likewise $A_j = |P_j|(1 + O(m^{-2}))$ by (3.29). Since the discs $B(p, R)$, $p \in Z_j$, are pairwise disjoint by (3.8), $|P_j| = 2\pi^2 - M_j\pi R^2$, so $\widetilde{A}_j = 2\pi^2 + O(m^{-2})|P_j| = 2\pi^2(1 + O(m^{-2}))$. The bounds on A_j follow from $\vartheta_j \le 1/(100\pi)$ for $m \ge m_0$. ☐

6.2. Poincaré and trace inequalities on the toral regions.

Lemma 6.2 (Uniform extension). *Let $Z = Z_j$ and $P = P_j$ be as in (3.6)–(3.7). There is a linear operator $E : H^1(P) \to H^1(\mathbb{T}_{\mathrm{fl}})$ with $Eu|_P = u$, $E(1) = 1$, which commutes with the action of $\mathcal{G}$ and satisfies*

$$\|\nabla Eu\|_{L^2(\mathbb{T}_{\mathrm{fl}})} \le \Lambda\,\|\nabla u\|_{L^2(P)} \qquad \textit{for all } u \in H^1(P),$$

with an absolute constant Λ.

Proof. For $p \in Z$ let $B_p := B(p, R)$ and $\mathcal{A}_p := B(p, 2R) \setminus \overline{B_p}$. By (3.8) distinct points of Z are at distance greater than $4R$, so the discs $B(p, 2R)$ are pairwise disjoint and each annulus $\mathcal{A}_p$ is contained in P. Let $\iota_p(x) := p + R^2(x-p)/|x-p|^2$ be the inversion in the circle ∂B_p; it is a conformal diffeomorphism of $\mathcal{A}_p$ onto $B_p \setminus \overline{B(p, R/2)}$ which is the identity on ∂B_p, and its Jacobian determinant on $\mathcal{A}_p$ is $(R/|x-p|)^4 \le 1$. Fix a smooth $\chi : [0, \infty) \to [0, 1]$ with $\chi = 0$ on $[0, \frac{1}{2}]$, $\chi = 1$ on $[\frac{3}{4}, \infty)$ and $|\chi'| \le 8$, and put $\chi_p(x) := \chi(|x-p|/R)$ and $\bar{u}_p := \operatorname{avg}_{\mathcal{A}_p} u$ (with respect to $dx\,dy$). Define

$$Eu := u \text{ on } P, \qquad Eu := \bar{u}_p + \chi_p \cdot \big((u - \bar{u}_p) \circ \iota_p\big) \text{ on } B_p \setminus \overline{B(p, R/2)}, \qquad Eu := \bar{u}_p \text{ on } \overline{B(p, R/2)}.$$

The three definitions agree on the common boundaries, since ι_p is the identity and $\chi_p = 1$ on ∂B_p, and $\chi_p = 0$ on $\partial B(p, R/2)$; hence $Eu \in H^1(\mathbb{T}_{\mathfrak{fl}})$, E is linear, $Eu|_P = u$ and $E(1) = 1$.

Let $w := (u - \bar{u}_p) \circ \iota_p$ on $B_p \setminus \overline{B(p, R/2)}$. By the conformal invariance of the Dirichlet integral, $\int |\nabla w|^2 = \int_{\mathcal{A}_p} |\nabla u|^2$; by the change of variables $x = \iota_p(y)$ and the bound on the Jacobian, $\int w^2 \le \int_{\mathcal{A}_p} (u - \bar{u}_p)^2$; and by the Poincaré inequality on the annulus $\{R < |x - p| < 2R\}$, whose constant scales with R^2, $\int_{\mathcal{A}_p} (u - \bar{u}_p)^2 \le C_{\mathcal{A}} R^2 \int_{\mathcal{A}_p} |\nabla u|^2$ with $C_{\mathcal{A}}$ the Poincaré constant of $\{1 < |x| < 2\}$. Since $|\nabla \chi_p| \le 8/R$,

$$\int_{B_p} |\nabla Eu|^2 \le 2 \int |\nabla w|^2 + 2 \cdot \frac{64}{R^2} \int w^2 \le \big(2 + 128\, C_{\mathcal{A}}\big) \int_{\mathcal{A}_p} |\nabla u|^2.$$

Summing over $p \in Z$, the annuli being pairwise disjoint subsets of P, $\|\nabla Eu\|^2_{L^2(\mathbb{T}_{\mathfrak{fl}})} \le (3 + 128\, C_{\mathcal{A}}) \|\nabla u\|^2_{L^2(P)}$. Finally each $\mathfrak{g} \in \mathcal{G}$ acts on $\mathbb{T}_{\mathfrak{fl}}$ by an affine isometry with $\mathfrak{g}(Z) = Z$, and $\mathfrak{g}(B_p) = B_{\mathfrak{g}p}$, $\mathfrak{g} \circ \iota_p = \iota_{\mathfrak{g}p} \circ \mathfrak{g}$, $\chi_{\mathfrak{g}p} \circ \mathfrak{g} = \chi_p$ and $\operatorname{avg}_{\mathcal{A}_{\mathfrak{g}p}} (u \circ \mathfrak{g}^{-1}) = \operatorname{avg}_{\mathcal{A}_p} u$; hence $E(u \circ \mathfrak{g}^{-1}) = (Eu) \circ \mathfrak{g}^{-1}$. $\square$

Lemma 6.3 (Poincaré inequality on the perforated torus)**.** *Let $P = P_j$ and $c_0 := \min(k, \ell)^2 / \Lambda^2$. For every $w \in H^1(P)^{\mathcal{G}}$,*

$$\int_P |\nabla w|^2 \, dx\, dy \;\ge\; c_0\, m^2 \int_P \big(w - \operatorname{avg}_P w\big)^2 \, dx\, dy.$$

Proof. Replacing w by $w - \operatorname{avg}_P w$, which is still $\mathcal{G}$-invariant and has the same gradient, we may assume $\int_P w = 0$. Let $W := Ew \in H^1(\mathbb{T}_{\mathfrak{fl}})^{\mathcal{G}}$ and $\overline{W} := \operatorname{avg}_{\mathbb{T}_{\mathfrak{fl}}} W$. Since $W = w$ on P and $\int_P w = 0$, $|\mathbb{T}_{\mathfrak{fl}}|\, \overline{W} = \int_{\mathbb{T}_{\mathfrak{fl}} \setminus P} W$, so by the Cauchy–Schwarz inequality $|\mathbb{T}_{\mathfrak{fl}}|^2\, \overline{W}^2 \le |\mathbb{T}_{\mathfrak{fl}} \setminus P| \int_{\mathbb{T}_{\mathfrak{fl}}} W^2$, that is $|\mathbb{T}_{\mathfrak{fl}}|\, \overline{W}^2 \le \vartheta_j \int_{\mathbb{T}_{\mathfrak{fl}}} W^2$. The function $W - \overline{W}$ has vanishing mean and is invariant under the translations by $(2X, 0)$ and $(0, 2Y)$, hence descends to the torus $\mathbb{R}^2 / (2X\mathbb{Z} \times 2Y\mathbb{Z})$, whose Laplace eigenvalues are $(\pi p/X)^2 + (\pi q/Y)^2 = 2m^2(k^2 p^2 + \ell^2 q^2)$, $p, q \in \mathbb{Z}$; the variational characterisation of the first nonzero eigenvalue on that torus gives

$$\begin{aligned}\int_{\mathbb{T}_{\mathfrak{fl}}} |\nabla W|^2 \;&\ge\; 2 \min(k, \ell)^2 m^2 \int_{\mathbb{T}_{\mathfrak{fl}}} \big(W - \overline{W}\big)^2 = 2 \min(k, \ell)^2 m^2 \Big(\int_{\mathbb{T}_{\mathfrak{fl}}} W^2 - |\mathbb{T}_{\mathfrak{fl}}|\, \overline{W}^2 \Big) \\ &\ge\; 2 \min(k, \ell)^2 m^2 (1 - \vartheta_j) \int_{\mathbb{T}_{\mathfrak{fl}}} W^2 \;\ge\; 2 \min(k, \ell)^2 m^2 (1 - \vartheta_j) \int_P w^2.\end{aligned}$$

With Lemma 6.2 and $2(1 - \vartheta_j) \ge 1$ this gives $\Lambda^2 \int_P |\nabla w|^2 \ge \min(k, \ell)^2 m^2 \int_P w^2$. $\square$

Corollary 6.4. *There is $c = c[k, \ell] > 0$ such that for $m \ge m_0$, every j and every $w \in H^1(\mathcal{T}[j])^{\mathcal{G}}$,*

$$\int_{\mathcal{T}[j]} |\nabla w|^2 \;\ge\; c\, m^2 \int_{\mathcal{T}[j]} \big(w - \operatorname{avg}_{\mathcal{T}[j]} w\big)^2.$$

Proof. Let w' be w transported to P_j by the chart of (3.28); it is $\mathcal{G}$-invariant. The mean $\operatorname{avg}_{\mathcal{T}[j]} w$ minimises $\int_{\mathcal{T}[j]} (w - \kappa)^2$ over constants κ. By (3.29) and Lemma 3.11, $dA \le (1 + \varepsilon_m)\, dx\, dy$ on $\mathcal{T}[j]$ and $\int_{P_j} |\nabla w'|^2 \, dx\, dy \le \frac{1 + \varepsilon_m}{1 - \varepsilon_m} \int_{\mathcal{T}[j]} |\nabla w|^2$, so with Lemma 6.3

$$\begin{aligned}\int_{\mathcal{T}[j]} \big(w - \operatorname{avg}_{\mathcal{T}[j]} w\big)^2 &\le \int_{\mathcal{T}[j]} \big(w - \operatorname{avg}_{P_j} w'\big)^2 \le (1 + \varepsilon_m) \int_{P_j} \big(w' - \operatorname{avg}_{P_j} w'\big)^2 \, dx\, dy \\ &\le \frac{1 + \varepsilon_m}{c_0 m^2} \int_{P_j} |\nabla w'|^2 \, dx\, dy \le \frac{(1 + \varepsilon_m)^2}{(1 - \varepsilon_m)\, c_0 m^2} \int_{\mathcal{T}[j]} |\nabla w|^2 \le \frac{4}{c_0 m^2} \int_{\mathcal{T}[j]} |\nabla w|^2,\end{aligned}$$

the last step because $\varepsilon_m = 3m^{-2} \le \frac{1}{3}$ for $m \ge 3$, whence $(1+\varepsilon_m)^2/(1-\varepsilon_m) \le \frac{8}{3}$. This is the claim with $c = c_0/4$. $\square$

Lemma 6.5 (Aggregate trace estimate)**.** *There is $K = K[k,\ell]$ such that for $m \ge m_0$, every j and every $u \in H^1(\mathcal{T}[j])^{\mathcal{G}}$,*

$$\sum_{\Gamma} \Big(\operatorname{avg}_{\Gamma} u - c_j\Big)^2 \;\le\; K \int_{\mathcal{T}[j]} |\nabla u|^2, \qquad c_j = \operatorname{avg}_{\mathcal{T}[j]} u,$$

the sum running over the M_j end circles Γ of the tunnels adjoining $\mathcal{T}[j]$.

Proof. Let u' be u transported to P_j. For $p \in Z_j$ let V_p be the Voronoi cell of p with respect to Z_j in $\mathbb{T}_{\mathrm{fl}}$. Since Z_j is a lattice (L_0, L_1, or $\mathsf{L}_0 \cup \mathsf{L}_1$), the cells V_p are translates of one convex polygon and tile $\mathbb{T}_{\mathrm{fl}}$ with disjoint interiors; since distinct points of Z_j are at distance at least $Y > 2R$, the closed disc $\overline{B(p,R)}$ lies in the interior of V_p. Thus $Q_p := V_p \setminus B(p,R)$ is a Lipschitz domain, all the Q_p are translates of one domain Q, and after rescaling by m the domain Q becomes one of two reference domains depending only on k and ℓ. On a bounded Lipschitz domain the trace theorem and the Poincaré inequality give, for the circle $\gamma = \partial B(p,R)$ with the mean taken with respect to the polar angle,

$$\Big(\operatorname{avg}_{\gamma} v - \operatorname{avg}_{Q_p} v\Big)^2 \le K_0 \int_{Q_p} |\nabla v|^2 \, dx\, dy \qquad \text{for all } v \in H^1(Q_p),$$

and both sides are invariant under dilations of the plane, so $K_0 = K_0[k,\ell]$ is independent of m. By Convention 3.10, $\operatorname{avg}_{\Gamma} u = \operatorname{avg}_{\gamma} u'$ for the end circle Γ over p. Summing over $p \in Z_j$ and using (3.29) with Lemma 3.11,

$$\sum_p \Big(\operatorname{avg}_{\gamma_p} u' - \operatorname{avg}_{Q_p} u'\Big)^2 \le K_0 \int_{P_j} |\nabla u'|^2 \, dx\, dy \le 3K_0 \int_{\mathcal{T}[j]} |\nabla u|^2.$$

On the other hand, by Jensen's inequality $|Q_p|\big(\operatorname{avg}_{Q_p} u' - c_j\big)^2 \le \int_{Q_p} (u' - c_j)^2$, and $|Q_p| = |P_j|/M_j \ge \pi^2/(2k\ell m^2)$, so by (3.29) and Corollary 6.4

$$\sum_p \Big(\operatorname{avg}_{Q_p} u' - c_j\Big)^2 \le \frac{2k\ell m^2}{\pi^2} \int_{P_j} (u' - c_j)^2 \, dx\, dy \le \frac{3k\ell m^2}{\pi^2} \int_{\mathcal{T}[j]} (u - c_j)^2 \le \frac{3k\ell}{\pi^2 c} \int_{\mathcal{T}[j]} |\nabla u|^2.$$

Since $(x+y)^2 \le 2x^2 + 2y^2$, the lemma follows with $K = 6K_0 + 6k\ell/(\pi^2 c)$. $\square$

6.3. **Mass on a channel.** The following lemma is the only place where the geometry of a tunnel enters beyond its conformal length. It is stated for an abstract channel, since it will be used in that form in Theorem 6.11; the hypotheses are satisfied by the tunnels of a stacked Clifford torus with $\rho = \tau_i \cosh t$ and $r = R$, as verified in the proof of Proposition 6.12.

Lemma 6.6 (Mass on a channel)**.** *There is an absolute constant C_0 with the following property. Let $a \ge 1$, $r > 0$ and $\varepsilon \in (0, \frac{1}{2}]$, let $\rho : [-a, a] \to (0, \infty)$ satisfy*

$$\rho(t) \;\le\; 2r\, e^{-(a-|t|)} \qquad (|t| \le a), \tag{6.2}$$

and let g be a Riemannian metric on $\mathbb{K}_a$ with $(1-\varepsilon)\rho^2 \widehat{\chi} \le g \le (1+\varepsilon)\rho^2 \widehat{\chi}$ and with $|\mathcal{K}^{\pm}|_g \le 2\pi r^2$, where $\mathcal{K}^{\pm} := \{\pm t \ge 0\}$ and $\Gamma_{\pm} := \{t = \pm a\}$. Then for every $u \in H^1(\mathbb{K}_a)$, writing $\bar{u}_{\pm} := \operatorname{avg}_{\Gamma_{\pm}} u$ and $\mathcal{E}_{\pm} := \int_{\mathcal{K}^{\pm}} |\nabla u|^2$,

$$\int_{\mathcal{K}^{\pm}} (u - \bar{u}_{\pm})^2 \le C_0 r^2 \, \mathcal{E}_{\pm}, \qquad \Big| \int_{\mathcal{K}^{\pm}} u^2 - |\mathcal{K}^{\pm}| \, \bar{u}_{\pm}^2 \Big| \le C_0 r^2 \Big(\mathcal{E}_{\pm} + |\bar{u}_{\pm}| \, \mathcal{E}_{\pm}^{1/2} \Big),$$

all integrals and norms on the left being taken with respect to g. One may take $C_0 = 81$.

Proof. We treat $\mathcal{K}^+ = \{0 \le t \le a\}$, the other half being identical. By density we may take u smooth. Put $\mathcal{E}_0 := \int_{\mathcal{K}^+}(u_t^2+u_\theta^2)\,dt\,d\theta$; by Lemma 3.11, the conformal invariance of the Dirichlet integral in dimension two and $\frac{1+\varepsilon}{1-\varepsilon} \le 3$, we have $\mathcal{E}_0 \le 3\mathcal{E}_+$, and $dA_g \le (1+\varepsilon)\rho^2\,dt\,d\theta \le \frac{3}{2}\rho^2\,dt\,d\theta$. By (6.2),

$$\rho^2 \le 4r^2 e^{-2(a-t)} \qquad (0 \le t \le a), \qquad \int_0^a \rho^2\,dt \le 2r^2. \tag{6.3}$$

Interior. For each θ, $(u(t,\theta)-u(a,\theta))^2 \le (a-t)\int_t^a u_s^2\,ds$, so by (6.3) and Fubini,

$$\int_0^a \big(u(t,\theta)-u(a,\theta)\big)^2\rho^2\,dt \le 4r^2\int_0^a u_s(s,\theta)^2\Big(\int_0^s (a-t)e^{-2(a-t)}\,dt\Big)\,ds \le r^2\int_0^a u_s(s,\theta)^2\,ds,$$

because $\int_0^\infty \sigma e^{-2\sigma}\,d\sigma = \frac{1}{4}$.

Boundary. Let $\mathcal{A} := [a-1,a]\times\mathbb{S}^1$ (recall $a \ge 1$) and $\bar u_{\mathcal{A}} := \operatorname{avg}_{\mathcal{A}} u$ with respect to $dt\,d\theta$, and put $v := u - \bar u_{\mathcal{A}}$. For $t \in [a-1,a]$, $v(a,\theta) = v(t,\theta) + \int_t^a v_s\,ds$; integrating in t over $[a-1,a]$ and applying the Cauchy–Schwarz inequality, $v(a,\theta)^2 \le 2\int_{a-1}^a v(t,\theta)^2\,dt + 2\int_{a-1}^a v_s(s,\theta)^2\,ds$. Integrating in θ and using the Poincaré inequality $\int_{\mathcal{A}} v^2 \le \int_{\mathcal{A}}|\nabla v|^2$ on the flat cylinder $[0,1]\times\mathbb{S}^1$, whose first nonzero Neumann eigenvalue is 1, we obtain $\int_0^{2\pi} v(a,\theta)^2\,d\theta \le 4\int_{\mathcal{A}}|\nabla u|^2\,dt\,d\theta \le 4\mathcal{E}_0$. Since $\bar u_+$ is the mean of $u(a,\cdot)$, $\int_0^{2\pi}(u(a,\theta)-\bar u_+)^2\,d\theta \le \int_0^{2\pi} v(a,\theta)^2\,d\theta \le 4\mathcal{E}_0$.

Conclusion. By $(x+y)^2 \le 2x^2+2y^2$, the interior estimate integrated in θ, and the boundary estimate with (6.3),

$$\int_{\mathcal{K}^+}(u-\bar u_+)^2\rho^2\,dt\,d\theta \le 2r^2\mathcal{E}_0 + 2\Big(\int_0^a \rho^2\,dt\Big)\int_0^{2\pi}\big(u(a,\theta)-\bar u_+\big)^2\,d\theta \le 18r^2\mathcal{E}_0,$$

hence $\int_{\mathcal{K}^+}(u-\bar u_+)^2\,dA \le 27r^2\mathcal{E}_0 \le 81r^2\mathcal{E}_+$. For the second inequality, expand $u^2 = \bar u_+^2 + 2\bar u_+(u-\bar u_+) + (u-\bar u_+)^2$ and estimate the middle term by the Cauchy–Schwarz inequality, using $|\mathcal{K}^+|^{1/2} \le (2\pi r^2)^{1/2} \le 3r$: $2|\bar u_+|\,|\mathcal{K}^+|^{1/2}\big(81r^2\mathcal{E}_+\big)^{1/2} \le 54r^2|\bar u_+|\mathcal{E}_+^{1/2}$. Both inequalities hold with $C_0 = 81$. ☐

6.4. **The weighted graph model.**

Definition 6.7. Let $\mathsf{G} = (V,E)$ be a finite connected graph without loops, each edge $e \in E$ carrying an orientation with tail e^- and head e^+. For weights $\widetilde A_v > 0$ ($v \in V$) and $C_e > 0$ ($e \in E$) set

$$\mu(\widetilde A, C) := \min\left\{\frac{\sum_{e\in E} C_e\big(c_{e^+}-c_{e^-}\big)^2}{\sum_{v\in V}\widetilde A_v c_v^2} \;:\; c \in \mathbb{R}^V\setminus\{0\},\ \sum_{v\in V}\widetilde A_v c_v = 0\right\},$$

the lowest Rayleigh quotient of G with vertex weights $\widetilde A_v$ and edge weights C_e on functions orthogonal to the constants. The quotient does not depend on the choice of orientations. The minimum exists, since the quotient is homogeneous and continuous on the unit sphere of the hyperplane $\sum_v \widetilde A_v c_v = 0$, and it is positive: the numerator vanishes only on the constant vectors, G being connected, and the only constant vector in that hyperplane is 0, since $\sum_v \widetilde A_v > 0$.

For $\mathsf{G} = P_N$ the path on $V = \{1,\dots,N\}$, with $e_i = \{i,i+1\}$ oriented from i to $i+1$, the numerator is $\sum_{i=1}^{N-1} C_i(c_i-c_{i+1})^2$; this is the case used in Theorems A and B.

Lemma 6.8 (Evaluation of the model)**.** *Let $\widetilde A_j$ and C_i be as in* (6.1) *and Definition 5.3, and let $C_N = 8\pi^2/(2-b_2[N])$. Then, for $m \ge m_0$,*

$$\mu(\widetilde A, C) = \frac{C_N}{2\pi^2}\,\lambda_1\big(L(P_N)\big)\big(1+O(m^{-2})\big) = 4\big(1+O(m^{-2})\big);$$

more precisely, writing $\widetilde{A}_j = 2\pi^2(1+\alpha_j)$ *and* $C_i = C_N(1+\beta_i)$ *with* $|\alpha_j|, |\beta_i| \le \eta$*, one has* $4(1-2\eta) \le \mu(\widetilde{A}, C) \le 4(1+4\eta)$ *whenever* $\eta \le \frac{1}{16}$*, so that in particular* $\mu(\widetilde{A}, C) \le 5$ *for* $m \ge m_0$*.*

Proof. By Proposition 4.4, $\lambda := \lambda_1(L(P_N)) = 2 - b_2[N]$, so $C_N\lambda/(2\pi^2) = 4$. By Lemma 6.1 and Proposition 5.4 the numbers α_j, β_i satisfy $|\alpha_j|, |\beta_i| \le \eta := Cm^{-2}$, and $\eta \le \frac{1}{16}$ for $m \ge m_0$. Write $\|c\|$ for the Euclidean norm, $\bar{c} := N^{-1}\sum_j c_j$, and note $c^\mathsf{T} L(P_N)c = \sum_i (c_i - c_{i+1})^2$.

Lower bound. Let $c \ne 0$ with $\sum_j \widetilde{A}_j c_j = 0$. Then $\sum_j c_j = -\sum_j \alpha_j c_j$, so $|\sum_j c_j| \le \eta\sqrt{N}\|c\|$ and $N\bar{c}^2 \le \eta^2\|c\|^2$. Since $c - \bar{c}\mathbf{1}$ is orthogonal to the constants, Lemma 4.2(ii) gives $c^\mathsf{T} Lc = (c-\bar{c}\mathbf{1})^\mathsf{T} L(c - \bar{c}\mathbf{1}) \ge \lambda\|c - \bar{c}\mathbf{1}\|^2 = \lambda(\|c\|^2 - N\bar{c}^2) \ge \lambda(1-\eta^2)\|c\|^2$. Hence

$$\frac{\sum_i C_i(c_i - c_{i+1})^2}{\sum_j \widetilde{A}_j c_j^2} \ge \frac{C_N(1-\eta)\lambda(1-\eta^2)}{2\pi^2(1+\eta)} = 4\,\frac{(1-\eta)(1-\eta^2)}{1+\eta} = 4(1-\eta)^2 \ge 4(1-2\eta).$$

Upper bound. Let $c^0 := c^{(1)}$ be the eigenvector of Lemma 4.2(ii), which satisfies $\sum_j c_j^0 = 0$ and $c^{0\mathsf{T}} Lc^0 = \lambda\|c^0\|^2$, and put $c := c^0 - \kappa\mathbf{1}$ with $\kappa := \sum_j \widetilde{A}_j c_j^0 / \sum_j \widetilde{A}_j$, so that $\sum_j \widetilde{A}_j c_j = 0$. Since $\sum_j \widetilde{A}_j c_j^0 = 2\pi^2\sum_j \alpha_j c_j^0$ and $\sum_j \widetilde{A}_j \ge 2\pi^2 N(1-\eta)$, we have $|\kappa| \le \eta\sqrt{N}\|c^0\|/(N(1-\eta)) \le 2\eta\|c^0\|/\sqrt{N}$, hence $\kappa^2\sum_j \widetilde{A}_j \le 4\eta^2 N^{-1}\|c^0\|^2 \cdot 2\pi^2 N(1+\eta) \le 12\pi^2\eta^2\|c^0\|^2$, and the differences $c_i - c_{i+1}$ are those of c^0. Since $\sum_j \widetilde{A}_j c_j^2 = \sum_j \widetilde{A}_j (c_j^0)^2 - \kappa^2\sum_j \widetilde{A}_j \ge 2\pi^2(1-\eta-6\eta^2)\|c^0\|^2 \ge 2\pi^2(1-2\eta)\|c^0\|^2$, we obtain

$$\mu(\widetilde{A}, C) \le \frac{C_N(1+\eta)\lambda\|c^0\|^2}{2\pi^2(1-2\eta)\|c^0\|^2} = 4\,\frac{1+\eta}{1-2\eta} \le 4(1+4\eta),$$

the last inequality because $(1+\eta) \le (1+4\eta)(1-2\eta)$ for $\eta \le \frac{1}{8}$. Finally $4(1+4\eta) \le 5$ for $\eta \le \frac{1}{16}$. □

6.5. **Graph decompositions.** We now isolate the properties of Σ which the reduction uses. All of them have been established above for stacked Clifford tori, and the verification is carried out in the proof of Proposition 6.12.

Definition 6.9 ($\mathcal{G}$-graph decomposition)**.** Let (Σ, g) be a closed connected surface, $\mathcal{G}$ a finite group of isometries of (Σ, g), and $\mathsf{G} = (V, E)$ a finite connected graph without loops, each edge $e \in E$ carrying an orientation with tail e^- and head e^+. Suppose given integers $M_e \ge 1$, real numbers $a_e \ge 1$ and $r_e > 0$, functions $\rho_e : [-a_e, a_e] \to (0, \infty)$, and constants $\varepsilon \in (0, \frac{1}{2}]$, $\eta \in [0, 1]$, $\mathsf{P} > 0$ and $\mathsf{K} > 0$. A *$\mathcal{G}$-graph decomposition of* Σ *modelled on* G, with these data, consists of compact subsurfaces with boundary $\mathcal{B}_v \subset \Sigma$ ($v \in V$), the *blocks*, and $\mathcal{K}_e^{(s)} \subset \Sigma$ ($e \in E$, $1 \le s \le M_e$), the *channels*, together with diffeomorphisms $\mathfrak{k}_e^{(s)} : \mathbb{K}_{a_e} \to \mathcal{K}_e^{(s)}$, subject to (D1)–(D6) below. We transport the coordinates (t, θ) of $\mathbb{K}_{a_e} = [-a_e, a_e] \times \mathbb{S}^1$ to $\mathcal{K}_e^{(s)}$ by $\mathfrak{k}_e^{(s)}$ and write $\Gamma_{e,\pm}^{(s)} := \mathfrak{k}_e^{(s)}(\{t = \pm a_e\})$ for the *end circles*, $\mathcal{K}_e^{(s),\pm} := \mathfrak{k}_e^{(s)}(\{\pm t \ge 0\})$ for the halves of a channel, and $\operatorname{avg}_\Gamma u := \frac{1}{2\pi}\int_0^{2\pi} u(\pm a_e, \theta)\,d\theta$ for $\Gamma = \Gamma_{e,\pm}^{(s)}$, in accordance with Convention 3.10.

- (D1) *Decomposition.* The blocks and the channels have pairwise disjoint interiors and their union is Σ; distinct channels are disjoint; $\mathcal{K}_e^{(s)} \cap \mathcal{B}_v$ equals $\Gamma_{e,-}^{(s)}$ if $v = e^-$, equals $\Gamma_{e,+}^{(s)}$ if $v = e^+$, and is empty otherwise; and $\partial\mathcal{B}_v$ is the disjoint union of the end circles of the channels of the edges incident to v.
- (D2) *Channels.* For all e and s,

$$(1-\varepsilon)\,\rho_e^2\,\widehat{\chi} \;\le\; (\mathfrak{k}_e^{(s)})^* g \;\le\; (1+\varepsilon)\,\rho_e^2\,\widehat{\chi} \quad \text{on } \mathbb{K}_{a_e}, \qquad \rho_e(t) \le 2r_e e^{-(a_e - |t|)} \quad (|t| \le a_e).$$

(D3) *Symmetry.* Every $\mathfrak{g} \in \mathcal{G}$ maps each block $\mathcal{B}_v$ onto itself and each channel $\mathcal{K}_e^{(s)}$ onto a channel $\mathcal{K}_e^{(s')}$ of the same edge, and in the channel coordinates $\mathfrak{g}$ acts by $(t, \theta) \mapsto (t, \pm\theta + \theta_0)$ for some $\theta_0 \in \mathbb{R}$ and some choice of sign.

(D4) *Areas.* Writing $A_v := |\mathcal{B}_v|$, one has $\big| |\mathcal{K}_e^{(s),\pm}| - \pi r_e^2 \big| \le \eta\, \pi r_e^2$ for all e, all s and both signs; and, setting $\Theta_v := \sum_{e \ni v} M_e \pi r_e^2$, the sum being over the edges incident to v, and $\widetilde{A}_v := A_v + \Theta_v$, one has $\Theta_v \le \frac{1}{2}\widetilde{A}_v$.

(D5) *Poincaré inequality on the blocks.* For every $v \in V$ and every $\mathcal{G}$-invariant $w \in H^1(\mathcal{B}_v)$, $\int_{\mathcal{B}_v} \big(w - \operatorname{avg}_{\mathcal{B}_v} w\big)^2 \le \mathsf{P} \int_{\mathcal{B}_v} |\nabla w|^2$.

(D6) *Aggregate trace inequality on the blocks.* For every $v \in V$ and every $\mathcal{G}$-invariant $u \in H^1(\mathcal{B}_v)$, $\sum_\Gamma \big(\operatorname{avg}_\Gamma u - \operatorname{avg}_{\mathcal{B}_v} u\big)^2 \le \mathsf{K} \int_{\mathcal{B}_v} |\nabla u|^2$, the sum running over the end circles $\Gamma \subset \partial\mathcal{B}_v$.

We further set

$$C_e := \frac{M_e \pi}{a_e}, \qquad \mathfrak{a} := \sum_{e \in E} M_e r_e^2, \qquad \mathfrak{b} := \sum_{e \in E} M_e^{1/2} r_e^2, \qquad \mathfrak{c} := \sum_{e \in E} M_e^{-1/2}, \tag{6.4}$$

and we call

$$\Xi := \varepsilon + \eta + \mathsf{P} + \mathfrak{b} + \mathfrak{c} \tag{6.5}$$

the *defect* of the decomposition.

Remark 6.10. Two consequences of (D3) will be used. First, a function which is constant on each block and which, on the channels of each edge e, depends only on t, is $\mathcal{G}$-invariant. Second, the restriction to a block of a $\mathcal{G}$-invariant function on Σ is $\mathcal{G}$-invariant, so that (D5) and (D6) apply to it. Note that (D3) does not require $\mathcal{G}$ to act transitively on the channels of an edge, and that no hypothesis is made on the metric of the blocks beyond (D4)–(D6).

6.6. The reduction theorem.

Theorem 6.11 (Reduction of the invariant sector)**.** *Let (Σ, g) carry a $\mathcal{G}$-graph decomposition modelled on $\mathsf{G} = (V, E)$ as in Definition 6.9, with defect Ξ, and set*

$$a_0 := \min_{v \in V} \widetilde{A}_v, \qquad a_1 := \max_{v \in V} \widetilde{A}_v, \qquad c_1 := \max_{e \in E} C_e.$$

There are constants $C^* > 0$ and $\Xi_0 > 0$, depending only on $|V|$, $|E|$, a_0, a_1, c_1 and K, such that if $\Xi \le \Xi_0$ then

$$\Big| \min\Big\{ \mathcal{R}(u) : u \in H^1(\Sigma)^{\mathcal{G}} \setminus \{0\},\ \textstyle\int_\Sigma u = 0 \Big\} - \mu(\widetilde{A}, C) \Big| \ \le\ C^* \,\Xi. \tag{6.6}$$

If moreover $\Sigma \subset \mathbb{S}^3$ is a closed embedded minimal surface, $\mathcal{G} < O(4)$ is generated by reflections in great spheres, and $\mu(\widetilde{A}, C) > 2 + C^*\Xi$, then $\lambda_1(\Sigma) = 2$. Inspection of the proof shows that C^* may be taken nondecreasing in a_1, c_1 and K and nonincreasing in a_0, and Ξ_0 with the opposite monotonicity, so that in applications it suffices to have a positive lower bound for the $\widetilde{A}_v$ and upper bounds for the $\widetilde{A}_v$, the C_e and K.

Proof. Throughout the proof C denotes a positive constant depending only on $|V|$, $|E|$, a_0, a_1, c_1 and K, which may change from line to line; for the duration of this proof this supersedes Convention 1.1. We write $\mu := \mu(\widetilde{A}, C)$, $\mathcal{E}_{\mathrm{blk}} := \sum_v \int_{\mathcal{B}_v} |\nabla u|^2$ and $\mathcal{E}_{\mathrm{ch}} := \sum_e \sum_s \int_{\mathcal{K}_e^{(s)}} |\nabla u|^2$ for the block and channel parts of the Dirichlet energy of a function u.

Preliminaries. Every edge is incident to exactly two vertices, so $\sum_v \Theta_v = 2\pi\mathfrak{a}$; with $\Theta_v \le \frac{1}{2}\widetilde{A}_v$ this gives

$$\mathfrak{a} \le \frac{1}{4\pi}\sum_v \widetilde{A}_v \le \frac{|V|a_1}{4\pi}, \qquad A_v \ge \tfrac{1}{2}\widetilde{A}_v \ge \tfrac{1}{2}a_0, \qquad |\Sigma| \ge \sum_v A_v \ge \tfrac{1}{2}|V|a_0, \tag{6.7}$$

and, since $1 \le M_e^{1/2} \le M_e$, also $\max_e r_e^2 \le \mathfrak{b} \le \mathfrak{a}$. Next,

$$\mu \;\le\; \bar{\mu} := \frac{8c_1|E|a_1^2}{a_0^3}. \tag{6.8}$$

Indeed, fix $v_0 \in V$, put $\kappa_0 := \widetilde{A}_{v_0}/\sum_w \widetilde{A}_w$ and $c := \mathbf{1}_{v_0} - \kappa_0\mathbf{1}$, so that $\sum_v \widetilde{A}_v c_v = 0$ and $c \ne 0$; the numerator in Definition 6.7 is $\sum_{e\ni v_0} C_e \le 2c_1|E|$, while, using $1-\kappa_0 = \sum_{w\ne v_0}\widetilde{A}_w/\sum_w \widetilde{A}_w \ge \frac{(|V|-1)a_0}{|V|a_1} \ge \frac{a_0}{2a_1}$, the denominator is at least $\widetilde{A}_{v_0}(1-\kappa_0)^2 \ge a_0^3/(4a_1^2)$. Finally, by (D2) and $|\mathcal{K}_e^{(s),\pm}| \le (1+\eta)\pi r_e^2 \le 2\pi r_e^2$, Lemma 6.6 applies on every channel with $a = a_e$, $r = r_e$, $\rho = \rho_e$; and Corollary 5.2 applies on every channel and gives, for $u \in H^1(\Sigma)$,

$$\int_{\mathcal{K}_e^{(s)}} |\nabla u|^2 \;\ge\; (1-2\varepsilon)\,\frac{\pi}{a_e}\,\delta_s^2, \qquad \delta_s := \operatorname{avg}_{\Gamma_{e,+}^{(s)}} u - \operatorname{avg}_{\Gamma_{e,-}^{(s)}} u, \tag{6.9}$$

since $\frac{1-\varepsilon}{1+\varepsilon} \ge 1-2\varepsilon$. We shall also use $\frac{1+\varepsilon}{1-\varepsilon} \le 1+4\varepsilon \le 3$ for $\varepsilon \le \frac{1}{2}$, and we agree that $\Xi_0 \le 1$, so that $\Xi^2 \le \Xi$ whenever $\Xi \le \Xi_0$.

Upper bound. Let $c \in \mathbb{R}^V$ realise μ, normalised by $\sum_v \widetilde{A}_v c_v^2 = 1$, so that $|c_v| \le \widetilde{A}_v^{-1/2} \le a_0^{-1/2}$; put $d_e := c_{e^+} - c_{e^-}$, so that $\sum_e C_e d_e^2 = \mu$ and $|d_e| \le 2a_0^{-1/2}$. Define $u := c_v$ on $\mathcal{B}_v$ and, on each channel of the edge e in its coordinates (t,θ),

$$u(t,\theta) := \frac{c_{e^-}+c_{e^+}}{2} + \frac{d_e}{2}\cdot\frac{t}{a_e}.$$

Then $u = c_{e^\pm}$ on $\{t = \pm a_e\}$, so u is continuous and piecewise smooth, hence in $H^1(\Sigma)$; it is $\mathcal{G}$-invariant by Remark 6.10; $|u| \le a_0^{-1/2}$ and $\operatorname{avg}_{\Gamma_{e,\pm}^{(s)}} u = c_{e^\pm}$.

On a channel of e the Dirichlet integral of u with respect to $\widehat{\chi}$ is $(d_e/2a_e)^2\cdot 2a_e\cdot 2\pi = \pi d_e^2/a_e$, so by (D2), Lemma 3.11 and the conformal invariance of the Dirichlet integral in dimension two, the energy $\mathcal{E}_s$ of u on that channel satisfies $(1-2\varepsilon)\pi d_e^2/a_e \le \mathcal{E}_s \le (1+4\varepsilon)\pi d_e^2/a_e \le 3\pi d_e^2/a_e$, whence, summing over s and over e,

$$\Big|\int_\Sigma |\nabla u|^2 - \mu\Big| \le 4\varepsilon\sum_e C_e d_e^2 = 4\varepsilon\mu \le 4\bar{\mu}\,\varepsilon, \qquad \sum_s \mathcal{E}_s \le 3C_e d_e^2 \le C. \tag{6.10}$$

By Lemma 6.6 with $\bar{u}_\pm = c_{e^\pm}$, then summing over s and using $\sum_s(\mathcal{E}_s^\pm)^{1/2} \le M_e^{1/2}(\sum_s \mathcal{E}_s)^{1/2} \le CM_e^{1/2}$ together with (D4),

$$\Big|\sum_s \int_{\mathcal{K}_e^{(s)}} u^2 - M_e\pi r_e^2\big(c_{e^-}^2 + c_{e^+}^2\big)\Big| \;\le\; C\big(\eta M_e r_e^2 + M_e^{1/2} r_e^2\big).$$

Since $\int_{\mathcal{B}_v} u^2 = A_v c_v^2$ and $\sum_e M_e \pi r_e^2(c_{e^-}^2 + c_{e^+}^2) = \sum_v \Theta_v c_v^2$, summing over e and using (6.7) gives

$$\Big|\int_\Sigma u^2 - 1\Big| = \Big|\int_\Sigma u^2 - \sum_v \widetilde{A}_v c_v^2\Big| \le C(\eta\mathfrak{a} + \mathfrak{b}) \le C\Xi. \tag{6.11}$$

Likewise, by the first inequality of Lemma 6.6 and the Cauchy–Schwarz inequality, $\big|\int_{\mathcal{K}_e^{(s),\pm}}(u-c_{e^\pm})\big| \le |\mathcal{K}_e^{(s),\pm}|^{1/2}(C_0 r_e^2\mathcal{E}_s^\pm)^{1/2} \le 3C_0^{1/2} r_e^2(\mathcal{E}_s^\pm)^{1/2}$, so that, summing over s as before,

$$\Big|\sum_s \int_{\mathcal{K}_e^{(s)}} u - M_e\pi r_e^2\big(c_{e^-}+c_{e^+}\big)\Big| \;\le\; C\big(\eta M_e r_e^2 + M_e^{1/2} r_e^2\big);$$

since $\int_{\mathcal{B}_v} u = A_v c_v$ and $\sum_v \widetilde{A}_v c_v = 0$,

$$\Big|\int_\Sigma u\Big| = \Big|\int_\Sigma u - \sum_v \widetilde{A}_v c_v\Big| \le C(\eta\mathfrak{a}+\mathfrak{b}) \le C\Xi. \tag{6.12}$$

Let $w := u - |\Sigma|^{-1}\int_\Sigma u$. Then $w \in H^1(\Sigma)^{\mathcal{G}}$, $\int_\Sigma w = 0$, $\int_\Sigma |\nabla w|^2 = \int_\Sigma |\nabla u|^2$, and by (6.7), (6.11) and (6.12), $\int_\Sigma w^2 = \int_\Sigma u^2 - |\Sigma|^{-1}(\int_\Sigma u)^2 \ge 1 - C\Xi$. Choosing Ξ_0 small enough that $C\Xi_0 \le \frac12$ we get $\int_\Sigma w^2 \ge \frac12$, so $w \ne 0$, and with (6.10),

$$\min\Big\{\mathcal{R}(u) : u \in H^1(\Sigma)^{\mathcal{G}}\setminus\{0\},\ \textstyle\int_\Sigma u = 0\Big\} \;\le\; \mathcal{R}(w) \;\le\; \frac{\mu+4\bar{\mu}\varepsilon}{1-C\Xi} \;\le\; \mu + C\Xi.$$

Lower bound. Let $u \in H^1(\Sigma)^{\mathcal{G}}$ with $\int_\Sigma u = 0$ and $\int_\Sigma u^2 = 1$, and put $\mathcal{E} := \int_\Sigma |\nabla u|^2 = \mathcal{R}(u)$. If $\mathcal{E} > \bar{\mu}$ then $\mathcal{E} > \mu \ge \mu - C^*\Xi$ and there is nothing to prove; so assume $\mathcal{E} \le \bar{\mu}$. Set $c_v := \operatorname{avg}_{\mathcal{B}_v} u$ and $d_e := c_{e^+} - c_{e^-}$. By Jensen's inequality $\sum_v A_v c_v^2 \le \sum_v \int_{\mathcal{B}_v} u^2 \le 1$, so by (6.7) $|c_v| \le A_v^{-1/2} \le (2/a_0)^{1/2}$, $|d_e| \le 2(2/a_0)^{1/2}$ and $\sum_e d_e^2 \le 8|E|/a_0$.

Step 1: energy. For a channel $\mathcal{K}_e^{(s)}$ set $\delta_\pm^{(s)} := \operatorname{avg}_{\Gamma_{e,\pm}^{(s)}} u - c_{e^\pm}$, so that δ_s of (6.9) equals $d_e + \delta_+^{(s)} - \delta_-^{(s)}$. By (D6), applied to $u|_{\mathcal{B}_{e^-}}$ and to $u|_{\mathcal{B}_{e^+}}$ and legitimate by Remark 6.10,

$$\sum_s \big(\delta_\pm^{(s)}\big)^2 \le \mathsf{K}\int_{\mathcal{B}_{e^\pm}} |\nabla u|^2 \le \mathsf{K}\mathcal{E}_{\mathrm{blk}} \le \mathsf{K}\bar{\mu}, \qquad \text{hence} \qquad \sum_s \big(\delta_+^{(s)} - \delta_-^{(s)}\big)^2 \le 4\mathsf{K}\mathcal{E}_{\mathrm{blk}}. \tag{6.13}$$

By the Cauchy–Schwarz inequality and $2xy \le x^2+y^2$,

$$M_e d_e^2 - M_e^{1/2}\big(d_e^2 + 4\mathsf{K}\mathcal{E}_{\mathrm{blk}}\big) \;\le\; \sum_{s=1}^{M_e}\delta_s^2 \;\le\; 2M_e d_e^2 + 8\mathsf{K}\mathcal{E}_{\mathrm{blk}},$$

so that, multiplying by $\pi/a_e = C_e/M_e$ and using (6.9),

$$\sum_s \int_{\mathcal{K}_e^{(s)}} |\nabla u|^2 \;\ge\; C_e d_e^2 - \frac{C_e}{M_e^{1/2}}\big(d_e^2 + 4\mathsf{K}\mathcal{E}_{\mathrm{blk}}\big) - 2\varepsilon\Big(2C_e d_e^2 + \frac{8\mathsf{K}C_e\mathcal{E}_{\mathrm{blk}}}{M_e}\Big).$$

Summing over e and using $\mathcal{E} \ge \mathcal{E}_{\mathrm{ch}}$, $\mathcal{E}_{\mathrm{blk}} \le \bar{\mu}$, $\sum_e d_e^2 \le 8|E|/a_0$, $C_e \le c_1$ and $M_e \ge 1$,

$$\mathcal{E} \;\ge\; \sum_{e\in E} C_e d_e^2 - C(\mathfrak{c}+\varepsilon) \;\ge\; \sum_{e\in E} C_e d_e^2 - C\Xi. \tag{6.14}$$

Step 2: mass. By (D5), since c_v is the mean of u on $\mathcal{B}_v$,

$$\int_{\mathcal{B}_v} u^2 = A_v c_v^2 + \int_{\mathcal{B}_v}(u-c_v)^2 \le A_v c_v^2 + \mathsf{P}\int_{\mathcal{B}_v}|\nabla u|^2.$$

On the channels of e, Lemma 6.6 gives, with $\bar{u}_\pm^{(s)} = c_{e^\pm} + \delta_\pm^{(s)}$ and $\mathcal{E}_s$ the energy of u on $\mathcal{K}_e^{(s)}$,

$$\sum_s \int_{\mathcal{K}_e^{(s)}} u^2 \le \sum_s \Big(\big|\mathcal{K}_e^{(s),-}\big|(\bar{u}_-^{(s)})^2 + \big|\mathcal{K}_e^{(s),+}\big|(\bar{u}_+^{(s)})^2\Big) + C_0 r_e^2 \sum_s \mathcal{E}_s + C_0 r_e^2 \sum_s \big(|\bar{u}_-^{(s)}| + |\bar{u}_+^{(s)}|\big)\mathcal{E}_s^{1/2}.$$

By (6.13), $\sum_s(\bar u_\pm^{(s)})^2 \le 2M_e c_{e\pm}^2 + 2\mathsf{K}\bar\mu \le CM_e$, whence by (D4) and the Cauchy–Schwarz inequality

$$\sum_s \big|\mathcal{K}_e^{(s),\pm}\big|(\bar u_\pm^{(s)})^2 \le (1+\eta)\pi r_e^2\Big(M_e c_{e\pm}^2 + 2|c_{e\pm}|M_e^{1/2}(\mathsf{K}\bar\mu)^{1/2} + \mathsf{K}\bar\mu\Big)$$
$$\le M_e\pi r_e^2 c_{e\pm}^2 + C\big(\eta M_e r_e^2 + M_e^{1/2} r_e^2\big),$$

while $C_0 r_e^2\sum_s \mathcal{E}_s \le C_0\bar\mu\, r_e^2$ and, again by Cauchy–Schwarz, $C_0 r_e^2\sum_s(|\bar u_-^{(s)}| + |\bar u_+^{(s)}|)\mathcal{E}_s^{1/2} \le C_0 r_e^2 (CM_e)^{1/2}\bar\mu^{1/2} \le CM_e^{1/2} r_e^2$. Summing over e and v and using $\sum_e M_e\pi r_e^2(c_{e-}^2 + c_{e+}^2) = \sum_v \Theta_v c_v^2$, $\mathcal{E}\le\bar\mu$ and (6.7),

$$1 = \int_\Sigma u^2 \;\le\; \sum_v \widetilde A_v c_v^2 + \mathsf{P}\bar\mu + C(\eta\mathfrak{a} + \mathfrak{b}) \;\le\; \sum_v \widetilde A_v c_v^2 + C\Xi. \tag{6.15}$$

Step 3: mean. By the first inequality of Lemma 6.6 and Cauchy–Schwarz, $\big|\int_{\mathcal{K}_e^{(s),\pm}}(u - \bar u_\pm^{(s)})\big| \le 3C_0^{1/2} r_e^2(\mathcal{E}_s^\pm)^{1/2}$, so that $\sum_s\big|\int_{\mathcal{K}_e^{(s),\pm}}(u-\bar u_\pm^{(s)})\big| \le 3C_0^{1/2} r_e^2 M_e^{1/2}\bar\mu^{1/2}$; moreover, by (D4), (6.13) and $\sum_s(\bar u_\pm^{(s)})^2 \le CM_e$,

$$\Big|\sum_s \big|\mathcal{K}_e^{(s),\pm}\big|\bar u_\pm^{(s)} - M_e\pi r_e^2 c_{e\pm}\Big| \le \pi r_e^2\Big|\sum_s \delta_\pm^{(s)}\Big| + \eta\pi r_e^2\sum_s\big|\bar u_\pm^{(s)}\big| \le C\big(M_e^{1/2} r_e^2 + \eta M_e r_e^2\big).$$

Since $\int_{\mathcal{B}_v} u = A_v c_v$, summing over e and v gives

$$\Big|\sum_v \widetilde A_v c_v\Big| = \Big|\int_\Sigma u - \sum_v \widetilde A_v c_v\Big| \le C(\eta\mathfrak{a} + \mathfrak{b}) \le C\Xi. \tag{6.16}$$

Step 4: conclusion. Put $\kappa := \sum_v \widetilde A_v c_v/\sum_v \widetilde A_v$, so that $|\kappa| \le C\Xi/(|V|a_0)$ by (6.16), and $c' := c - \kappa\mathbf{1}$. Then $\sum_v \widetilde A_v c_v' = 0$, the differences $c_{e+}' - c_{e-}' = d_e$ are unchanged, and by (6.15)

$$\sum_v \widetilde A_v (c_v')^2 = \sum_v \widetilde A_v c_v^2 - \kappa^2\sum_v \widetilde A_v \;\ge\; 1 - C\Xi - C\Xi^2 \;\ge\; 1 - C\Xi.$$

By Definition 6.7, $\sum_e C_e d_e^2 \ge \mu\sum_v \widetilde A_v(c_v')^2$, an inequality which holds trivially if $c' = 0$; hence $\sum_e C_e d_e^2 \ge \mu(1 - C\Xi) \ge \mu - C\bar\mu\Xi$, which with (6.14) gives $\mathcal{E} \ge \mu - C\Xi$. Together with the upper bound this proves (6.6).

The last assertion. By Lemma 2.1, $\lambda_1(\Sigma) \le 2$. Suppose $\lambda_1(\Sigma) < 2$ and let u be a first eigenfunction. By Lemma 2.2, u is $\mathcal{G}$-invariant; since also $\int_\Sigma u = 0$ and $u \ne 0$, (6.6) gives $2 > \lambda_1(\Sigma) = \mathcal{R}(u) \ge \mu(\widetilde A, C) - C^*\Xi > 2$, a contradiction. Hence $\lambda_1(\Sigma) = 2$. □

6.7. **Reduction of the invariant sector to the path graph.**

Proposition 6.12. *For $m \ge m_0$,*

$$\Big|\ \min\Big\{\mathcal{R}(u) : u \in H^1(\Sigma)^{\mathcal{G}}\setminus\{0\},\ \textstyle\int_\Sigma u = 0\Big\} - \mu(\widetilde A, C)\ \Big| \;\le\; \frac{C^*}{m}$$

for a constant $C^* = C^*[N,k,\ell]$.

Proof. We verify the hypotheses of Definition 6.9 for Σ and then apply Theorem 6.11. We take for G the path P_N on $V = \{1,\dots,N\}$, with the edge $e_i = \{i, i+1\}$ oriented from i to $i+1$, and we set

$$\mathcal{B}_j := \mathcal{T}[j], \qquad \mathcal{K}_{e_i}^{(s)} := \mathcal{K}_i^{(s)}, \qquad M_{e_i} := M = k\ell m^2,$$
$$a_{e_i} := a_i, \qquad r_{e_i} := R, \qquad \rho_{e_i}(t) := \tau_i\cosh t,$$

with $\varepsilon := \varepsilon_m = 3m^{-2}$ and with the coordinates, end circles and halves of (3.28).

(D1). This is the decomposition (3.28): the pieces have pairwise disjoint interiors, the tunnels are pairwise disjoint, and $\mathcal{K}_i^{(s)}$ meets $\mathcal{T}[i]$ exactly along $\Gamma_{i,-}^{(s)}$ and $\mathcal{T}[i+1]$ exactly along $\Gamma_{i,+}^{(s)}$. By Lemma 3.3 and (3.7), $\partial\mathcal{T}[j]$ is the disjoint union of the M_j circles lying over $\partial B(p,R)$, $p \in Z_j$, that is of the end circles of the tunnels adjoining $\mathcal{T}[j]$.

(D2). The two-sided bound is the first line of (3.29), and $\varepsilon_m \leq \frac{1}{2}$ for $m \geq m_0$. Since $\tau_i \cosh a_i = R$ by (3.13) we have $\tau_i \leq 2Re^{-a_i}$, whence $\rho_{e_i}(t) = \tau_i \cosh t \leq \tau_i e^{|t|} \leq 2Re^{-(a_i - |t|)}$. Finally $a_i \geq 1$ for $m \geq m_0$, by (3.14) and Corollary 4.5.

(D3) is the statement established at the beginning of this section.

(D4). By Lemma 6.1, $|\mathcal{K}_i^{(s),\pm}| = \pi R^2(1 + O(m^{-2}))$, so $\eta = Cm^{-2} \leq 1$ for $m \geq m_0$. The vertex j is incident to one edge if $j \in \{1, N\}$ and to two otherwise, so $\sum_{e \ni j} M_e \pi R^2 = M_j \pi R^2$, which is the quantity Θ_j of (6.1); hence A_j, Θ_j and $\widetilde{A}_j$ are those of (6.1). By (6.1) and Lemma 6.1, $\Theta_j = 2\pi^2 \vartheta_j \leq \pi/50 \leq \frac{1}{2}\pi^2 \leq \frac{1}{2}\widetilde{A}_j$.

(D5) is Corollary 6.4, with $\mathsf{P} = 1/(cm^2)$ and $c = c[k,\ell]$.

(D6) is Lemma 6.5, with $\mathsf{K} = K[k,\ell]$.

The conductance $C_{e_i} = M\pi/a_i$ is the C_i of Definition 5.3, and the weighted graph of Definition 6.7 is the weighted path occurring in Lemma 6.8, so that $\mu(\widetilde{A}, C)$ is the same quantity in both statements. By Lemma 6.1 we have $\pi^2 \leq a_0 \leq a_1 \leq 3\pi^2$, and $c_1 \leq C[N]$ by Proposition 5.4; these bounds, together with $|V| = N$, $|E| = N - 1$ and $\mathsf{K} = K[k,\ell]$, depend only on N, k and ℓ, so the constants C^* and Ξ_0 of Theorem 6.11 do as well. Finally ε, η and P are $O(m^{-2})$, while, by (3.4),

$$\mathfrak{b} = (N-1)M^{1/2}R^2 = \frac{(N-1)\sqrt{k\ell}}{100\,\ell^2 m}, \qquad \mathfrak{c} = (N-1)M^{-1/2} = \frac{N-1}{\sqrt{k\ell}\,m},$$

so that $\Xi = O(m^{-1})$ with a constant depending only on N, k, ℓ. Enlarging m_0 so that $\Xi \leq \Xi_0$, Theorem 6.11 gives the assertion. $\square$

Remark 6.13. The weights $\widetilde{A}_j = A_j + \Theta_j$ are the areas of the tori *before* the discs are removed. Steps 2 and 3 in the proof of Theorem 6.11, and the corresponding computations in its upper bound, show that, to the order $O(m^{-1})$, the halves of the tunnels restore exactly the removed discs, both in the L^2 norm and in the mean; no term of the size ϑ_j of the removed area survives. This is what makes the reduction two-sided with error $O(m^{-1})$ and is used in Theorem B; for Theorem A alone a one-sided bound with a loss of order ϑ_j would suffice.

7. Proofs of Theorems A and B

Proof of Theorem B. Part (i) is Proposition 4.4. Part (ii): the first identity is Proposition 5.4; the identity $C_N \lambda_1(L(P_N))/(2\pi^2) = 4$ is the computation

$$\frac{C_N}{2\pi^2}\,\lambda_1\big(L(P_N)\big) = \frac{1}{2\pi^2} \cdot \frac{8\pi^2}{2 - b_2[N]} \cdot \big(2 - b_2[N]\big) = 4,$$

in which $\lambda_1(L(P_N)) = 2 - b_2[N]$ is Lemma 4.1 together with the identification $b_2[N] = \lambda_{\max}(A(P_{N-1}))$ of Proposition 4.4; the evaluation $\mu(\widetilde{A}, C) = 4 + O(m^{-2})$ is Lemma 6.8. Part (iii) follows from $\lambda_1(L(P_N)) = 2(1 - \cos(\pi/N)) = \pi^2 N^{-2} - \frac{\pi^4}{12}N^{-4} + O(N^{-6})$ and from Proposition 5.4. $\square$

Proof of Theorem A. By Lemma 2.1, $\lambda_1(\Sigma) \leq 2$. Suppose $\lambda_1(\Sigma) < 2$ and let u be a first eigenfunction. The surface Σ is closed, embedded, minimal and $\mathcal{G}$-invariant, and $\mathcal{G}$ is generated by reflections in great spheres by Lemma 3.1; hence Lemma 2.2 gives $u \in H^1(\Sigma)^{\mathcal{G}}$, while $\int_\Sigma u = 0$ and $\mathcal{R}(u) = \lambda_1(\Sigma) < 2$. On the other hand, by Proposition 6.12 and Lemma 6.8, for $m \geq m_0[N,k,\ell]$ every $\mathcal{G}$-invariant function with vanishing mean satisfies

$$\mathcal{R}(u) \;\geq\; \mu(\widetilde{A}, C) - \frac{C^*}{m} \;\geq\; 4\big(1 - Cm^{-2}\big) - \frac{C^*}{m} \;\geq\; 3,$$

a contradiction. Hence $\lambda_1(\Sigma) = 2$.

For the second assertion, note that the Laplacian of Σ commutes with the isometric action of $\mathcal{G}$, so the closed subspace $L^2(\Sigma)^{\mathcal{G}}$ of $\mathcal{G}$-invariant functions is spanned by $\mathcal{G}$-invariant eigenfunctions, and the smallest nonzero eigenvalue occurring there is $\min\{\mathcal{R}(u) : u \in H^1(\Sigma)^{\mathcal{G}} \setminus \{0\},\ \int_\Sigma u = 0\}$ by the variational characterisation applied in that subspace, the kernel being spanned by the constants. By Proposition 6.12 and Lemma 6.8 this minimum equals $\mu(\widetilde{A}, C) + O(m^{-1}) = 4 + O(m^{-1})$.

Finally, by Corollary 3.8 every surface produced by [22, Theorem 6.49] with $m \geq m_0$ is a stacked Clifford torus of type (N, k, ℓ, m), after m_0 has been enlarged to exceed the threshold of that lemma; hence Yau's conjecture holds for all these surfaces. □

Remark 7.1. The proof uses about the symmetry group only that it is generated by reflections in great spheres, that it preserves each torus, and that it acts on functions without sign. Whether or not Σ admits further isometries exchanging the two sides of $\mathbb{T}$, which for $k \neq \ell$ it does not [22, Remark 1.2] and for $k = \ell$ is not decided in [22], plays no role; in particular the argument is the same for all $N \geq 2$. The coordinate functions of $\mathbb{R}^4$, which span an eigenspace for the eigenvalue 2, are not $\mathcal{G}$-invariant, in accordance with the value $4 + O(m^{-1})$ of the invariant sector.

8. The balancing–spectrum identity

8.1. **The mechanism.** Consider a chain of N tori of area $2\pi^2$ joined by families of tunnels of total conductance C. The mode which is constant on each torus has Rayleigh quotient $\frac{C}{2\pi^2}\,\lambda_1(L(P_N)) \sim \frac{C}{2\pi^2}\,\pi^2 N^{-2}$. Were C bounded independently of N, this would tend to 0 and, by Lemma 2.1, long chains would violate Yau's conjecture. What excludes this is the balancing. The linearised balancing system says that the vector of waist ratios is a positive eigenvector of $A(P_{N-1})$ with eigenvalue b_2, hence its Perron vector (Lemma 4.3, Proposition 4.4); the incidence identity (Lemma 4.1) makes the spectral gap of the chain equal to $2 - b_2$; and the conformal length (3.14) of the tunnels is proportional to $2 - b_2$, so their conductance is proportional to $(2 - b_2)^{-1}$. The cancellation is exact for every N and requires no evaluation of b_2. Equivalently, by (3.10), the waist radii increase with N: the necks of a long chain are forced to be fatter, by exactly the right amount. Remark 4.6 adds that the vector of heights of the tori is the Fiedler vector of P_N, which is why the mode governed by the gap, and not another block-constant mode, is the one realised by the configuration.

8.2. **The constant** 4.

Remark 8.1. The surviving constant is the coefficient of the Jacobi operator (1.2), and this has a direct explanation at the level of the linearised equation which does not pass through the balancing conditions; we describe it informally, and it is not used in the proofs. Away from the waists, each torus of Σ is a normal graph over a parallel torus of a function solving the linearised minimal surface equation $\mathcal{J}_{\mathbb{T}}\varphi = \Delta\varphi + 4\varphi = 0$ to leading order, and so are the ends of the tunnels; the normal coordinate of Σ with respect to $\mathbb{T}$ is therefore, to leading order, a function u with $\Delta u + 4u = 0$ away from the waists, whose total area is $O(\tau_1^2 M)$, and integration by parts gives $\mathcal{R}(u) \to 4 = |A|^2 + \mathrm{Ric}(\nu, \nu)$. The same explanation applies to doublings, where the corresponding mode is odd for the side-exchanging involution, and to doublings of the equatorial two-sphere, where the analogous constant is $|A|^2 + \mathrm{Ric}(\nu,\nu) = 0 + 2 = 2$ and is realised exactly by the coordinate function vanishing on the base sphere. What Sections 4–6 add is that the value 4 is attained, for every N, by the mode governed by the *spectral gap* of the chain, which is the statement needed for Theorem A: a mode with quotient close to 4 would not by itself exclude lower block-constant modes in a long chain.

8.3. **Trees.** The identity $\lambda_1(L(P_N)) = 2 - \lambda_{\max}(A(P_{N-1}))$ is a property of paths. We record the general statement.

Proposition 8.2. *Let T be a finite tree with at least one edge, $L(T)$ its combinatorial Laplacian and $\mathrm{Line}(T)$ its line graph. Then*

$$\lambda_1\big(L(T)\big) = 2 + \lambda_{\min}\big(A(\mathrm{Line}(T))\big), \qquad \lambda_{\max}\big(L(T)\big) = 2 + \lambda_{\max}\big(A(\mathrm{Line}(T))\big),$$

and $\lambda_1(L(T)) = 2 - \lambda_{\max}(A(\mathrm{Line}(T)))$ holds if and only if T is a path.

Proof. Let $|D|$ be the unsigned $|V| \times |E|$ incidence matrix of T. Then $|D||D|^{\mathsf{T}} = \mathrm{diag}(\deg) + A(T) =: Q(T)$ is the signless Laplacian and $|D|^{\mathsf{T}}|D| = 2I + A(\mathrm{Line}(T))$, by the same entry computation as in Lemma 4.1 but without signs. Since T is bipartite, with parts $V_1 \sqcup V_2$, the diagonal matrix S with entries $+1$ on V_1 and -1 on V_2 satisfies $SA(T)S = -A(T)$, because every edge joins V_1 to V_2; hence $SQ(T)S = L(T)$ and $Q(T)$ and $L(T)$ have the same spectrum (see [2] for these standard facts). As T is connected, $\ker L(T)$ is one-dimensional, so $Q(T)$ has exactly $|V| - 1 = |E|$ nonzero eigenvalues counted with multiplicity; these are the nonzero eigenvalues of $|D|^{\mathsf{T}}|D|$, an $|E| \times |E|$ matrix, which therefore has no zero eigenvalue. Thus the spectrum of $2I + A(\mathrm{Line}(T))$ is the spectrum of $L(T)$ with 0 removed, which gives the two identities.

For the last statement, the identity in question holds if and only if $\lambda_{\min}(A(\mathrm{Line}(T))) = -\lambda_{\max}(A(\mathrm{Line}(T)))$. We use the following classical fact: for a connected graph H with adjacency matrix A and spectral radius ρ, the number $-\rho$ is an eigenvalue of A if and only if H is bipartite. Indeed, if $Av = -\rho v$ with $v \neq 0$, then $\rho|v| = |Av| \le A|v|$ entrywise, so $\langle A|v|, |v|\rangle \ge \rho\|v\|^2$ and $|v|$ maximises the Rayleigh quotient of A; hence $A|v| = \rho|v|$, $|v|$ is the Perron vector, which is positive since A is irreducible, and equality $|Av| = A|v|$ forces, for every vertex i, all v_j with $j \sim i$ to have the common sign $-\operatorname{sgn} v_i$. Thus $\operatorname{sgn} v$ is a proper 2-colouring of H. Conversely, if H is bipartite with Perron vector w and S is the sign matrix of the bipartition, then $A(Sw) = -SAw = -\rho\, Sw$. Now the line graph of a connected graph is connected. If some vertex of T has degree at least 3, three edges at that vertex are pairwise adjacent and form a triangle in $\mathrm{Line}(T)$, which is then not bipartite, and the identity fails. If every vertex of T has degree at most 2, then T, being a connected tree, is a path P_N, and $\mathrm{Line}(P_N) = P_{N-1}$ is bipartite, so the identity holds. Hence the identity holds exactly when T is a path. $\square$

The algebraic content of the mechanism of Theorem B is the following statement about the operator $D^{\mathsf{T}}D$ of a tree. As the proof records, $L(T) = D_\sigma D_\sigma^{\mathsf{T}}$ for every orientation σ; we write $\lambda_1(L(T))$ and $\lambda_{\max}(L(T))$ for its smallest nonzero and its largest eigenvalue.

Proposition 8.3 (The balancing operator of a tree)**.** *Let T be a finite tree with $n \ge 2$ vertices and edge set E, let σ be an orientation of its edges, let D_σ be the corresponding incidence matrix as in Section 4, and set $A_\sigma := 2I - D_\sigma^{\mathsf{T}} D_\sigma$, so that $(A_\sigma)_{ee} = 0$, while for $e \neq f$ one has $(A_\sigma)_{ef} = 0$ if e and f are not adjacent and, if they meet at a vertex v, $(A_\sigma)_{ef} = +1$ if v is the head of one of them and the tail of the other, and $(A_\sigma)_{ef} = -1$ otherwise. Then:*

(i) *$D_\sigma^\intercal D_\sigma$ is positive definite and its spectrum, with multiplicities, is the spectrum of $L(T)$ with the eigenvalue 0 removed; in particular every eigenvalue κ of $D_\sigma^\intercal D_\sigma$ satisfies $\kappa \geq \lambda_1(L(T))$.*

(ii) *A_σ has nonnegative entries if and only if T is a path and σ orients it consistently, and in that case $A_\sigma = A(\mathrm{Line}(T))$.*

(iii) *If A_σ has nonnegative entries and some $\tau \in \mathbb{R}^E$ with positive entries satisfies $D_\sigma^\intercal D_\sigma \tau = \kappa\tau$, then $\kappa = \lambda_1(L(T))$ and τ is a positive multiple of the Perron vector of $A(\mathrm{Line}(T))$.*

(iv) *For the star $T = K_{1,3}$ with all three edges oriented away from the centre one has $D_\sigma^\intercal D_\sigma = 2I + A(K_3)$, whose spectrum is $\{4,1,1\}$; its eigenvectors with positive entries are exactly the positive multiples of $(1,1,1)$, and the corresponding eigenvalue is $\kappa = 4 = \lambda_{\max}(L(K_{1,3}))$.*

Proof. The entries of $D_\sigma^\intercal D_\sigma$ are computed as in Lemma 4.1: $(D_\sigma^\intercal D_\sigma)_{ee} = 2$, while for $e \neq f$ meeting at v the only nonzero term of $\sum_w D_{we} D_{wf}$ is the one at $w = v$, which equals -1 if v is the head of one of e, f and the tail of the other and $+1$ if it is the head of both or the tail of both; this gives the stated form of A_σ. Likewise $(D_\sigma D_\sigma^\intercal)_{vw}$ equals the degree of v if $v = w$, equals $(+1)(-1) = -1$ if $\{v, w\}$ is an edge and vanishes otherwise, whatever the orientation; so $D_\sigma D_\sigma^\intercal = L(T)$.

(i) Since T is connected, $\ker D_\sigma^\intercal$ consists of the constant vectors, so $\operatorname{rank} D_\sigma = n - 1 = |E|$ and the $|E| \times |E|$ matrix $D_\sigma^\intercal D_\sigma$ is positive definite. The nonzero eigenvalues of $D_\sigma D_\sigma^\intercal$ and $D_\sigma^\intercal D_\sigma$ coincide with multiplicities, and $L(T)$ has exactly $n - 1 = |E|$ of them; these are therefore all the eigenvalues of $D_\sigma^\intercal D_\sigma$.

(ii) Suppose $A_\sigma \geq 0$ entrywise. If some vertex v had degree at least 3, then among three edges at v two would be both incoming or both outgoing at v, and the corresponding entry of A_σ would be -1. Hence every vertex of T has degree at most 2, and T, being a connected tree, is a path; and each vertex of degree 2 must be the head of one of the two edges meeting there and the tail of the other, so σ orients the path consistently. Conversely, for a consistently oriented path every pair of adjacent edges meets at a vertex which is the head of one and the tail of the other, so $(A_\sigma)_{ef} = 1$ exactly when e and f are adjacent, that is $A_\sigma = A(\mathrm{Line}(T))$.

(iii) By (ii), $A_\sigma = A(\mathrm{Line}(T))$ with $\mathrm{Line}(T) = P_{n-1}$, which is connected, so A_σ is nonnegative and irreducible. From $D_\sigma^\intercal D_\sigma \tau = \kappa\tau$ we get $A_\sigma \tau = (2 - \kappa)\tau$ with $\tau > 0$, so by the Perron–Frobenius theorem τ is, up to a positive multiple, the Perron vector and $2 - \kappa = \lambda_{\max}(A_\sigma)$. Hence κ is the smallest eigenvalue of $D_\sigma^\intercal D_\sigma$, which is $\lambda_1(L(T))$ by (i).

(iv) The centre is the tail of all three edges, so $(A_\sigma)_{ef} = -1$ for every pair of distinct edges, that is $A_\sigma = -A(K_3)$ and $D_\sigma^\intercal D_\sigma = 2I + A(K_3)$. The spectrum of $A(K_3)$ is $\{2, -1, -1\}$, whence the spectrum $\{4, 1, 1\}$; the eigenspace for 4 is spanned by $(1,1,1)$, while the eigenspace for 1 consists of the vectors with vanishing sum, none of which has positive entries. Finally the spectrum of $L(K_{1,3})$ is $\{0, 1, 1, 4\}$, so $\lambda_{\max}(L(K_{1,3})) = 4$. □

Remark 8.4. Proposition 8.2 delimits the reach of the mechanism of Theorem B, and Proposition 8.3 isolates its algebraic core. In the linear model which produced (3.9), blocks v of a base surface with Jacobi constant q and area $|\Sigma_0|$ carry heights h_v, necks e carry waists τ_e and a common leading conformal length a, and the two balancing relations read $q|\Sigma_0|h = 2\pi M\, D_\sigma \tau$ and $D_\sigma^\intercal h = 2a\tau$, where D_σ is the incidence matrix of T with each edge oriented from the lower to the upper block, as in Lemma 4.1 for the chain. Hence $D_\sigma^\intercal D_\sigma \tau = \kappa\tau$ with $\kappa = q|\Sigma_0|a/(\pi M)$, the total conductance is $C = M\pi/a = q|\Sigma_0|/\kappa$, and the lowest block-constant Rayleigh quotient is $q\,\lambda_1(L(T))/\kappa$. Now Proposition 8.3(ii)–(iii) applies: the matrix $A_\sigma = 2I - D_\sigma^\intercal D_\sigma$ has nonnegative entries exactly when every block has at most one neighbour on each side, that is when T is a path traversed consistently; then the positivity of τ forces $\kappa = \lambda_1(L(T))$, and the quotient is q. For a star with three blocks on the same side of the centre one finds instead, by Proposition 8.3(iv), $\kappa = 4 = \lambda_{\max}(L(K_{1,3}))$ and the quotient $q\lambda_1/\lambda_{\max} = q/4$, which for the Clifford torus is $1 < 2$; such a configuration is of course not realisable by parallel tori. The exact cancellation is therefore an intrinsically one-dimensional phenomenon.

Problem 8.5. Let a closed embedded minimal surface be produced by gluing congruent blocks of a base surface Σ_0 with constant $q = |A|^2 + \mathrm{Ric}(\nu, \nu)$ along the edges of a finite graph G, the waist parameters being determined by linearised balancing conditions. Is the lowest Rayleigh quotient of the block-constant modes always $q\,\lambda_1(L(G))/\kappa + o(1)$, with κ the eigenvalue of the balancing vector as in Remark 8.4, and for which graphs G and sign structures can such surfaces exist? The desingularisations of intersecting Clifford tori of [12], in which several sheets meet along a common locus, are natural test cases; for several of these families the surfaces are determined by their symmetries and topology [11], so that the combinatorial datum G is intrinsic.

8.4. **First eigenfunctions.** Yau's conjecture implies through [18] that a minimal embedding of a closed surface in $\mathbb{S}^3$ is by first eigenfunctions, and hence area bounds of the type of [23]. Theorem A places the surfaces of [22] among the known minimal embeddings by first eigenfunctions, a class enlarged considerably by the equivariant eigenvalue optimisation of [15] and by [16]. Our argument gives the eigenvalue but not the eigenspace.

Problem 8.6. Is the first eigenspace of a stacked Clifford torus spanned by the restrictions of the coordinate functions of $\mathbb{R}^4$?

The argument of Section 6 shows that no $\mathcal{G}$-invariant first eigenfunction exists; a positive answer would require controlling the sectors of the other irreducible representations of $\mathcal{G}$, on which the reflection lemma gives no information.

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