GAUDUCHON METRICS AND STABILITY

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1. Stable vector bundles

I) Algebraic case.

Let X be a projective manifold of dimension d polarised by a very ample divisor H. Let $\mathcal{E}$ be a torsion-free coherent sheaf of rank r over X. Then the number $c_1(\mathcal{E}).H^{d-1}$ is called the *degree of $\mathcal{E}$ relative to H* and it is denoted by $\deg_H(\mathcal{E})$. The number $\deg_H(\mathcal{E})/r$ is called the *slope of $\mathcal{E}$ relative to H* and it is denoted by $\mu_H(\mathcal{E})$.

DEFINITION 1. *(Mumford-Takemoto) A torsion-free coherent sheaf $\mathcal{E}$ over X is H-stable if for every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < rk(\mathcal{F}) < rk(\mathcal{E})$, we have*

$$\mu_H(\mathcal{F}) < \mu_H(\mathcal{E}).$$

A torsion-free coherent sheaf $\mathcal{E}$ over X is H-semistable if for every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < rk(\mathcal{F}) < rk(\mathcal{E})$, we have

$$\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E}).$$

Let now $\mathcal{E}$ be a rank r topological (smooth) vector bundle and let $\mathcal{M}(\mathcal{E})$ be the set of all algebraic structures in $\mathcal{E}$. Unfortunately, this set $\mathcal{M}(\mathcal{E})$ is too big, i.e. it has not an algebraic structure (as an algebraic variety). This was the reason of introducing stable vector bundles. We have

THEOREM 1. *(Gieseker-Maruyama) Let X be an algebraic surface ($d = 2$) and let H be a very ample divisor on X. Then, the set $\mathcal{M}_H(\mathcal{E})$ of all H-stable vector bundles with topological support $\mathcal{E}$ has an algebraic structure (as an algebraic scheme), which is a coarse moduli space for the induced functor.*

If X is a surface (compact, connected, complex manifold of dimension two) the topological (smooth) classification of complex vector bundles was given by Wu: the diffeomorphism classes of complex topological vector bundles are parametrized by the set

$$\{(r, c_1, c_2) \mid r \text{ an integer} \geq 2, c_1 \in H^2(X, \mathbb{Z}),\ c_2 \in H^4(X, \mathbb{Z})\,\}.$$

In the case of the complex projective space there is only one notion of stability given by a hyperplane H of the projective space. Generally, the stability is relative as the following example shows:

EXAMPLE 1. *Let $X = \mathbb{F}_1$ be the ruled surface over $\mathbb{P}_1$ with invariant $e = 1$. Let E be a rank-2 algebraic vector bundle and denote by S_E the set of ample line bundles H on X such that E is H-stable. Then, there exist two rank-2 algebraic vector bundles on X E and E' with the Chern classes $c_1 = (1, 0)$, $c_2 = 5$ such that $S_E \neq S_{E'}$. The proof is quite involved and it is based on some difficult results of Qin and Aprodu-Brinzanescu.*

II) Kaehler case.

Let X be a compact Kaehler manifold of dimension d with Kaehler metric g and let ω be its Kaehler form; it is a real positive closed $(1,1)$-form on X. Let $\mathcal{E}$ be a torsion-free coherent sheaf over X and let $c_1(\mathcal{E})$ be its real first Chern class; it is represented by a real closed $(1,1)$-form on X. We can define the em degree of $\mathcal{E}$ *relative to g* by

$$\deg_g(\mathcal{E}) := \int_X c_1(\mathcal{E}) \wedge \omega^{d-1},$$

and the *slope of $\mathcal{E}$ relative to g* by

$$\mu_g(\mathcal{E}) := \deg_g(\mathcal{E})/rk(\mathcal{E})$$

.

Obviously, one can extend the definition of g-stability and of g-semistability to this case as in the projective case.

III) General case.

More generally, let X be now a compact complex manifold of dimension d. We can endow the manifold X with a hermitian metric.

DEFINITION 2. *A hermitian metric g on X is called a Gauduchon metric if its "Kaehler form" ω_g satisfies the following second order differential equation:*

$$\partial\overline{\partial}\omega_g^{d-1} = 0.$$

The following result of Gauduchon shows that on any compact complex manifold there are a lot of Gauduchon metrics:

THEOREM 2. *(Gauduchon) Any hermitian metric on X is conformally equivalent to a Gauduchon metric. If $d \geq 2$, then this Gauduchon metric is unique up to a positive factor.*

Let g be a Gauduchon metric on X. By an idea of Hitchin, if L is a holomorphic line bundle over X, the *degree of L with respect to ω_g* can be defined by

$$\deg_g(L) := \int_X (i/2\pi)F \wedge \omega_g^{d-1},$$

where F is the curvature of any hermitian connection on L compatible with $\overline{\partial}_L$. Since any two such forms differ by a $\partial\overline{\partial}$-exact form, $\deg_g(L)$ is independent of the choice of connection. If $d\omega_g = 0$, then $\deg_g(L)$ is the usual topological degree defined above, but in general, $\deg_g(L)$ is not a topological invariant.

Having defined the degree of holomorphic line bundles, the degree of a torsion-free coherent sheaf $\mathcal{E}$ is defined by

$$\deg_g(\mathcal{E}) := \deg_g(\det(\mathcal{E}))$$

and the definition of g-stability (respectively g-semistability) can be repeated verbatim.

2. Hermitian-Einstein connections

Let X be a compact complex manifold and let g be a hermitian metric on X. Let ω_g be its "Kaehler form" and let us denote by Λ_g the L^2-adjoint (the contraction with the metric) of the multiplication operator $x \to x \wedge \omega_g$. If (E,h) is a hermitian rank r-vector bundle over the hermitian manifold (X,g) and, if A is a unitary connection in E, then F_A denotes the curvature form of the connection A.

DEFINITION 3. *Let (X,g) be a hermitian compact complex manifold and let (E,h) be a hermitian rank r-vector bundle on it. A unitary connection A in E is called weakly Hermitian-Einstein (w.H-E) if F_A has type $(1,1)$ and if it exists a real function ϕ_A on X such that*

$$\Lambda_g F_A = i\phi_a . Id_E.$$

If $\phi_A = C_A$ is a constant function, then A is called Hermitian-Einstein (H-E) and this constant is called the Einstein constant of A.

This concept has been introduced by Kobayashi as a generalisation of the notion of a Kaehler-Einstein metric. If a connection A is w.H-E then, $\overline{\partial}_A$ is integrable, so it defines a holomorphic structure E_A in E.

THEOREM 3. *(Kobayashi-Luebke) Let (E,h) be an H-E vector bundle over a compact complex manifold X endowed with a Gauduchon metric g. Then E is g-semistable and (E,h) is a direct sum*

$$(E,h) = (E_1,h_1) \oplus ... \oplus (E_k,h_k)$$

of g-stable H-E vector bundles with the same constant C as (E,h).

REMARK 1. *The converse has been conjectured independently by Kobayashi and Hitchin. The Kobayashi-Hitchin correspondence relates the complex geometry concept of "stable holomorphic vector bundles" to the differential geometry concept of "H-E connection".*

THEOREM 4. *Let $X,g)$ be a compact complex manifold endowed with a Gauduchon metric. A holomorphic vector bundle E over X is g-stable if and only if it allows a hermitian metric h such that the associated Chern connection A_h is H-E and irreducible. This metric is unique up to a constant factor.*

The existence of H-E connections in stable bundles was proved by Donaldson for projective surfaces and later for projective manifolds, by Uhlenbeck and Yau for Kaehler manifolds, by Buchdahl for Gauduchon surfaces and finally, by Li and Yau for general Gauduchon manifolds.

Very interesting applications of moduli spaces of stable vector bundles to the theory of 4-dimensional real manifolds were obtain by Donaldson-Kronheimer and Friedman-Morgan.

3. Vector bundles on non-Kähler elliptic surfaces

Let $X \xrightarrow{\pi} B$ be a minimal non-Kähler elliptic surface with B a smooth curve of genus g. It is well-known that $X \xrightarrow{\pi} B$ is a quasi-bundle over the base B, that is, all the smooth fibres are isomorphic to a fixed elliptic curve E and the singular fibres (in a finite number) are multiples of elliptic curves (see, for example, [7], [5]) .

Let V be a holomorphic rank-2 vector bundle on X, with fixed $c_1(V) = c_1 \in NS(X)$ and $c_2(V) = c_2 \in \mathbb{Z}$. Now, we fix also the determinant line bundle of V, denoted by $\delta = det(V)$. It defines an involution on the relative Jacobian $J(X) = B \times E^*$ of X:

$$i_\delta : J(X) \to J(X),\ (b, \lambda) \to (b, \delta_b \otimes \lambda^{-1}),$$

where δ_b denotes the restriction of δ to the fibre $E_b = \pi^{-1}(b)$, which has degree zero (see Lemma 2.2 in [10]). Taking the quotient of $J(X)$ by this involution, each fibre of p_1 becomes $E^*/i_\delta \cong \mathbb{P}^1$ and the quotient $J(X)/i_\delta$ is isomorphic to a ruled surface $\mathbb{F}_\delta$ over B. Let $\eta : J(X) \to \mathbb{F}_\delta$ be the canonical map.

Now, let $\mathcal{F}$ be an analytic coherent sheaf over a surface X of rank $r > 0$, with Chern classes $c_1(\mathcal{F})$ and $c_2(\mathcal{F})$. The *discriminant* $\Delta(\mathcal{F})$ is defined by

$$\Delta(\mathcal{F}) := \frac{1}{r}\left(c_2(\mathcal{F}) - \frac{r-1}{2r}c_1^2(\mathcal{F})\right).$$

For a non-algebraic surface X, $a \in NS(X)$ and r a positive integer we can define the following rational positive number (see [5], [9], [3])

$$m(r, a) := -\frac{1}{2r} max\{\Sigma_1^r (a/r - \mu_i)^2,\ \ \mu_i \in NS(X)\ \ with\ \ \Sigma_1^r \mu_i = a\}.$$

The main existence result of holomorphic rank-2 vector bundles over non-Kähler elliptic surfaces is the following (see [10]):

THEOREM 5. *(Brinzanescu-Moraru) Let X be a minimal non-Kähler elliptic surface over a smooth curve B of genus g and fix a pair (c_1, c_2) in $NS(X) \times \mathbb{Z}$. Set $m_{c_1} := m(2, c_1)$ and denote $\overline{c}_1$ the class of c_1 in $NS(X)$ modulo $2NS(X)$; moreover, let $e_{\overline{c}_1}$ be the invariant of the ruled surface $\mathbb{F}_{\overline{c}_1}$ determined by $\overline{c}_1$. Then, there exists a holomorphic rank-2 vector bundle on X with Chern classes c_1 and c_2 if and only if*

$$\Delta(2, c_1, c_2) \geq (m_{c_1} - d_{\overline{c}_1}/2),$$

where $d_{\overline{c}_1} := (e_{\overline{c}_1} + 4m_{c_1})/2$. Note that both $d_{\overline{c}_1}$ and $(m_{c_1} - d_{\overline{c}_1}/2)$ are non-negative numbers. Furthermore, if

$$(m_{c_1} - d_{\overline{c}_1}/2) \leq \Delta(2, c_1, c_2) < m_{c_1},$$

then the corresponding vector bundles are non-filtrable.

The main tool to study vector bundles on any elliptic surface X is by taking restrictions to the smooth fibres. Note that if X is non-Kähler, then the restriction of any line bundle on X to a smooth fibre of π always has degree zero; see [10]. For a rank two vector bundle V over X, its restriction to a generic fibre of π is semistable; more precisely, its restriction to a fibre $\pi^{-1}(b)$ is unstable on at most an isolated set of points $b \in B$ and, these isolated points are called the *jumps* of the bundle. Furthermore, there exists a divisor S_V in the relative Jacobian $J(X) = B \times E^*$ of X, called the *spectral curve* or *cover* of the bundle, that encodes the isomorphism class of the bundle over each fibre of π. The spectral curve is the following divisor

$$S_V := \Sigma_1^k(\{x_i\} \times E^*) + \overline{C},$$

where $\overline{C}$ is a bisection of $J(X)$ (i.e. $\overline{C}.E^* = 2$) and $x_1, x_2, ..., x_k$ are points in B that correspond to the jumps of V.

The spectral curve is constructed by using a twisted Fourier - Mukai transform . For more details, see [11], Section 3, Theorem 3.1, [16] and [14].

By construction, the spectral curve S_V of the bundle V is invariant by the involution i_δ of $J(X)$, and descends to the quotient $\mathbb{F}_\delta$; in fact, it is a pullback via η of a divisor on $\mathbb{F}_\delta$ of the form

$$\mathcal{G}_V := \Sigma_1^k f_i + A,$$

where f_i is the fibre of the ruled surface over the point x_i and A is a section of the ruling such that $\eta^* A = \overline{C}$. The divisor $\mathcal{G}_V$ is called *the graph of* V.

Fix a rank-2 vector bundle V on a minimal non-Kähler elliptic surface X and let δ be its determinant line bundle; there exists a sufficient condition on the spectral cover of V that ensures its stability (see [12]):

PROPOSITION 1. *Suppose that the spectral cover of V includes an irreducible bisection $\overline{C}$ of $J(X)$. Then V is irreducible, and hence it is also stable with respect to any Gauduchon metric.*

Let X be a minimal non-Kähler elliptic surface and consider a pair (c_1, c_2) in $NS(X) \times \mathbb{Z}$. We fix a Gauduchon metric on X. For a fixed line bundle δ on X with $c_1(\delta) = c_1$, let $\mathcal{M}_{\delta,c_2}$ be the moduli space of stable (with respect to the fixed Gauduchon metric) holomorphic rank-2 vector bundles with invariants $\det(V) = \delta$ and $c_2(V) = c_2$. Note that, for any $c_1 \in NS(X)$, one can choose a line bundle δ on X such that

$$c_1(\delta) \in c_1 + 2NS(X) \quad \text{and} \quad m(2, c_1) = -\frac{1}{2}(c_1(\delta)/2)^2;$$

moreover, if there exist line bundles a and δ' such that $\delta = a^2\delta'$, then there is a natural isomorphism between the moduli spaces $\mathcal{M}_{\delta,c_2}$ and $\mathcal{M}_{\delta',c_2}$, defined by $V \to a \otimes V$.

This moduli space can be identified, via the Kobayashi - Hitchin correspondence, with the moduli space of gauge-equivalence classes of Hermitian - Einstein connections in the fixed differentiable rank-2 vector bundle determined by δ and c_2 (see, for example, [15], [28]). In particular, if the determinant δ

is the trivial line bundle $\mathcal{O}_X$, then there is a one-to-one correspondence between $\mathcal{M}_{\mathcal{O}_X,c_2}$ and the moduli space of $SU(2)$-instantons, that is, anti-selfdual connections.

We can define the map

$$G : \mathcal{M}_{\delta,c_2} \to Div(\mathbb{F}_\delta)$$

that associates to each stable vector bundle its graph in $Div(\mathbb{F}_\delta)$, called the *graph map*. In [4], [30], the stability properties of vector bundles on Hopf surfaces were studied by analysing the image and the fibres of this map.

For the general case, the moduli spaces $\mathcal{M}_{\delta,c_2}$ are studied by Brînzănescu - Moraru in [12].

THEOREM 6. *(Brinzanescu-Moraru) Let $X \xrightarrow{\pi} B$ be a non-Kähler elliptic surface and let $\mathcal{M}_{\delta,c_2}$ be defined as above. Then:*

(i) There are necessary and sufficient conditions such that $\mathcal{M}_{\delta,c_2}$ is nonempty (see Theorem 5).

(ii) If $c_2 - c_1^2/2 > g-1$ (g is the genus of B), the moduli space $\mathcal{M}_{\delta,c_2}$ is smooth on the open dense subset of regular bundles (a regular bundle is a vector bundle for which its restriction to any fibre has its automorphism group of the smallest dimension).

(iii) The generic fibre of the graph map $G : \mathcal{M}_{\delta,c_2} \to Div(\mathbb{F}_\delta)$ is a Prym variety (for Prym varieties, see [31]*).*

(iv) Let $\mathbb{P}_{\delta,c_2}$ be the set of divisors in $\mathbb{F}_\delta$ of the form $\Sigma_1^k f_i + A$, where A is a section of the ruling and the f_i's are fibres of the ruled surface, that are numerically equivalent to $\eta_*(B_0) + c_2 f$. For $c_2 \geq 2$, the graph map is surjective on $\mathbb{P}_{\delta,c_2}$. For $c_2 < 2$, see [12].

(v) Explicit descriptions of the the singular fibres of G are given, see [12].

Special results on the moduli space $\mathcal{M}_{\delta,c_2}$ in the case of primary Kodaira surfaces are given in [2].

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