Open problems in LCK geometry

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Conformal structures in geometry
– On the occasion of Liviu Ornea's 60th birthday –

Zoom, July 16, 2020

Prologue

The many facets of Liviu Ornea

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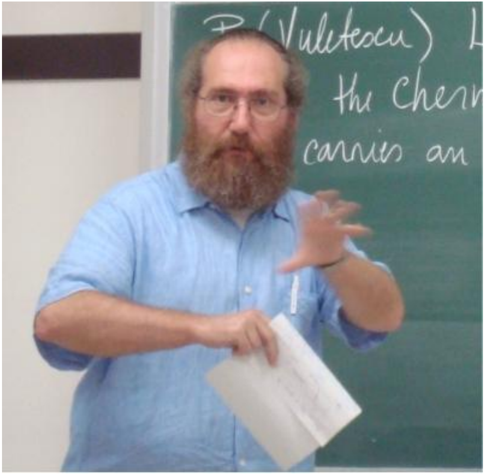


Professor

The many facets of Liviu Ornea

Theater critic

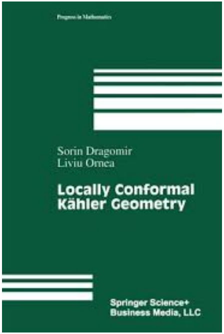
The many facets of Liviu Ornea

Columnist at *Observatorul Cultural*

The many facets of Liviu Ornea

Actor (*Aferim*, 2015)

The many facets of Liviu Ornea



Founder of LCK geometry in Romania

Part I

LCK structures:

Definition and first properties

Kähler structures

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- odd degree Betti numbers are even : $b_{2k+1} \in 2\mathbb{Z}$.
- even degree Betti numbers are non-zero : $b_{2k} > 0$, $\forall k \leq \dim_{\mathbb{C}} M$.

LCK structures

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No Kähler metrics on simple complex manifolds, e.g. $S^1 \times S^{2n-1}$ *for* $n \geq 2$.

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Remark

The Lee form $\theta = 0 \iff \omega$ is Kähler.

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θ exact $\Longrightarrow$ g is globally conformally Kähler (GCK).

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Theorem (Vaisman)

If (M, J) satisfies the $\partial\bar{\partial}$-Lemma (in particular if it carries a Kähler metric), then any LCK metric on (M, J) is GCK.

Part II

Examples

Examples of LCK manifolds

Example (Compact complex manifolds admitting LCK metrics)

- Hopf manifolds: $\mathbb{Z}$-quotients of $\mathbb{C}^n \setminus \{0\}$, diffeomorphic to $S^1 \times S^{2n-1}$ (Vaisman)

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- LCK metrics with potential: If $(\tilde{M}, J)$ has a positive PSH function φ which is automorphic wrt the action of a discrete co-compact group Γ of holomorphisms of $\tilde{M}$ (i.e. $\gamma^*\varphi = c_\gamma \varphi$, $\forall \gamma \in \Gamma$), then $\omega := i\frac{\partial\bar{\partial}\varphi}{\varphi}$ defines an LCK structure on $M := \tilde{M}/\Gamma$, with Lee form $\theta = -\mathrm{d} \ln \varphi$.

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Part III

Conjectures and open problems

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Remark

No topological obstruction for the existence of (strict) LCK metrics is known, except $b_1 > 0$.

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Conjecture (Ornea)

A product of two compact complex manifolds carries an LCK metric if and only if they are both of Kähler type.

Partial results

Theorem (Tsukada, 1999)

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If M_1 and M_2 are compact complex manifolds and M_1 is of Kähler type, then $M_1 \times M_2$ has an LCK metric $\iff$ M_2 is of Kähler type.

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- M_2 has an LCK metric with potential (Istrati)
- M_2 has a complex curve C (Ornea-Verbitsky)
- The restriction of the LCK structure to $M_1 \times C$ is GCK, so the restriction of the Lee form to $M_1 \times C$ is exact, so by Künneth θ is cohomologous to a pullback $p_2^*\theta_2$.

□

Thank you for your attention!

Happy birthday, Liviu!